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III. *On the Laws of Crystâlline Reflexion and Refraction.* By JAMES MAC CULLAGH, *Fellow of Trinity College, Dublin.*

Read 9th January, 1837.

WHEN a ray of light, which has been polarised in a given plane, suffers reflexion and refraction at the surface of a transparent medium, the rays into which it is divided are found to be polarised in certain other planes; and it becomes a question to determine the positions of these planes, as well as the relative intensities of the different rays; or, in theoretical language, to find the direction and magnitude of the reflected and refracted vibrations, supposing those of the incident vibration to be given. The transparent medium may be either a singly refracting substance, such as glass, or a doubly refracting crystal like Iceland spar. When the medium is of the first kind, the problem is comparatively simple, being, in fact, nothing more than a particular case of the problem which we have to consider when the medium is supposed to be of the second kind. In the progress of knowledge it was natural that the simpler question should be first attended to; and accordingly Fresnel, during his brief and brilliant career, found time to solve it. But the general problem, relative to doubly refracting media, had not been attempted by any one, when, in the year 1834, my thoughts were turned to the subject. I then recollected a conclusion to which I had been led some years before, and which, on this occasion, proved of essential service to me. Being fond of geometrical constructions, I amused myself, when I first became acquainted with Fresnel's theories, by throwing his algebraical expressions, whenever I could, into a geometrical form; and treating in this way the well-known formulæ in which he has embodied his solution of the problem just alluded to, I obtained a remarkable result, which gave me the first view of the principle that I have since employed under the name of the *principle of the equivalence of vibrations.* In order to state this result briefly, I will take leave to introduce a new term for expressing a right line drawn parallel to the plane of polarisation of a ray, and perpendicular to the direction of the ray itself. Calling such a right line the *transversal* of the polarised ray, I found, from the formulæ of Fresnel, that when polarised light falls upon a singly refracting medium, the transversals of the incident, of the reflected, and of the refracted rays are all parallel to the same plane, which is the plane of polarisation of the refracted ray; and that the magnitudes of the vibrations, or the greatest excursions of the ethereal molecules, in the incident and the reflected rays, are to each other inversely as the sines of the angles which the respective transversals of those rays make with the transversal of the refracted ray. I was struck by the strong analogy which these relations among the transversals bore to the composition of forces or of small vibrations in mechanics; but it happened unfortunately, that, in the theory of Fresnel, the vibrations of light were supposed to take place, not in the direction of the transversals, but perpendicular to them, so that there was no physical circumstance to support the analogy, there being no motion in the direction of the transversals; while, on the other hand, no such analogy existed among the vibrations themselves in the directions which Fresnel had assigned to them. It was therefore with some interest that I afterwards learned, upon the publication of the tenth volume of the Memoirs of the Institute, that M. Cauchy* had actually inferred, from mechanical principles, that the vibrations of polarised light are in the direction of the transversals; but this inference was to be received with caution, as being contrary to the hypothesis of Fresnel; and besides, I had in the mean time contrived a way of adapting my analogy, in some degree, to that hypothesis, by supposing *areas* to be compounded instead of vibrations; so that I hesitated which of the two opinions to prefer. Taking, however, the opinion of M. Cauchy as that which fell in more naturally with the aforesaid analogy, I was led to the conclusion, that the vibration in the refracted ray is probably the resultant of the incident and reflected vibrations; and I saw that if this principle were true for singly refracting media, it should also, from its very nature, be true, when properly generalised, for doubly refracting crystals; so that in such crystals the resultant of the two refracted vibrations would be the same, both in length and direction, as the resultant of the incident and reflected vibrations.

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* Mémoires de l'Institut, tome x. p. 304.

This was the principle of *equivalent vibrations.* But I had no sooner begun to regard it as probable, than an objection started up against it. In the case of a ray ordinarily refracted out of a rarer into a denser medium, the magnitude of the refracted vibration, as deduced from this principle, was greater than that which came out from the theory of Fresnel, in the proportion of the sine of the angle of incidence to the sine of the angle of refraction. Consequently, assuming with Fresnel, that the ether is more dense in the denser medium, the law of the preservation of *vis viva* was violated.

There was another embarrassment which I felt in my early efforts to find out the laws of crystalline reflexion. Taking for granted the hypothesis of Fresnel, that the density of the ether in an ordinary medium is inversely as the square of its refractive index, I was at a loss what hypothesis to make, in this respect, for doubly refracting crystals, wherein the refractive index changes with the direction of the ray. For the density, being independent of direction, could not be conceived to vary with the refractive index. About two years ago, I got over this difficulty by supposing the density of the ether to be the same in all media.* At the same time I was compelled to employ the principle of equivalent vibrations, in order to have a sufficient number of conditions, though for a while I overlooked the perfect agreement which now subsisted between this principle and the law of *vis viva;* it happened, in fact, that the new hypothesis of a constant density made the *vis viva* of the refracted ray exactly the same as in the theory of Fresnel.†

But to see why it was necessary to assume the principle of equivalent vibrations, we must observe, that when a polarised ray is incident on a crystal, there are four things to be determined, namely, the direction and magnitude of the reflected vibration, and the magnitudes of the two refracted vibrations. Hence we must have four conditions, or we must have relations affording so many equations. But the hypotheses of Fresnel, by which he solved the problem of reflexion for ordinary media, afford only three conditions. We will state his hypotheses at length:—

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* This hypothesis is maintained by Mr. Challis; and certainly it falls in extremely well with the astronomical phenomenon of the aberration of light.—See, on this subject, Professor Lloyd's Report on Physical Optics, Fourth Report of the British Association for the Advancement of Science, pp. 311, 313.

† See hereafter, p. 42, *note.*

1st. The vibrations of polarised light are in the plane of the wave, and *perpendicular* to the plane of polarisation.

2nd. The density of the ether is inversely as the square of the refractive index of the medium.

3rd. The *vis viva* is preserved.

4th. The vibrations parallel to the separating surface of two media are equivalent; that is, the refracted vibration parallel to the surface is the resultant of the incident and reflected vibrations parallel to the same.

We see that the fourth hypothesis gives two conditions, and the law of *vis viva* gives a third.

Let us now take the more general principle of equivalent vibrations, in place of the fourth hypothesis of Fresnel, altering the first hypothesis in the way that we have shown to be necessary in order to suit that principle, and making the ethereal density constant. Then, if we retain the law of *vis viva*, our new hypotheses will be these:—

1st. The vibrations of polarised light are in the plane of the wave, and *parallel* to the plane of polarisation; which may be expressed in a word, by saying that the vibrations are *transversal*, according to the peculiar sense in which I use the term.

2nd. The density of the ether is the same in all bodies as in *vacuo*.

3rd. The *vis viva* is preserved.

4th. The vibrations in two contiguous media are equivalent; that is, the resultant of the incident and reflected vibrations is the same, both in length and direction, as the resultant of the refracted vibrations.

It is evident that the last hypothesis affords three equations, by resolving the vibrations parallel to three axes of coordinates; and the law of *vis viva* supplies a fourth equation. Thus we have the requisite number of conditions.

The hypotheses that we have last enumerated are those which will be employed in the present paper. They have been made to include the law of *vis viva*, because I lately found that this law must necessarily accompany the rest; but at first I neglected it, and even made considerable progress without it; for, by the help of another hypothesis, I obtained formulæ which represented such experiments as I was aware of at the time. This other hypothesis I took up from reading an article by M. Cauchy in the Bulletin des Sciences Mathématiques,[*] in which he arrives, by a peculiar process, at the formulæ of Fresnel for the case of ordinary reflexion. The hypotheses which he chiefly employs are relations among certain quantities called pressures; and it was such a relation that I adopted instead of the law of *vis viva*. I supposed that, at the confines of two media, the pressure on the separating surface, in a direction perpendicular to the plane of incidence, ought to be the same, whether it be considered as resulting from the vibrations in the first medium or in the second. This hypothesis I conceived to be true in general, because I found it to be true for ordinary media; but I could never assign any better reason for it. Combining it, however, with the principle of equivalent vibrations, I deduced several expressions for uniaxal crystals, and among others a formula for the polarising angles in different azimuths of the plane of reflexion. When this formula was compared with the experiments of Sir David Brewster* on the polarising angles of Iceland spar, the accordance was so satisfactory, as to leave no doubt upon my mind that I had arrived at the true formula for these angles; and though the truth of the conclusion did not allow me to argue that the premises were true, yet the presumption in their favour was very strong, insomuch that, upon remarking, as I did soon after, that the law of *vis viva* harmonized with my other hypotheses, I did not think it worth while† to try what would be the consequence of using this law, instead of the relation which I had put in its place. In this state of my theory, I gave an account of it at the meeting of the British Association‡ in Dublin, in August, 1835; and the leading steps and results were afterwards published in a letter to Sir David Brewster.§

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[*] Sur la Réfraction et la Réflexion de la Lumière, Bulletin des Sci. Math. Juillet, 1830. In this paper the vibrations of polarised light must be supposed perpendicular to the plane of polarisation, though the paper was published immediately after the author had promulged the contrary opinion. The latter opinion, which I adopted from him because it harmonized with my analogy before mentioned, he has formally renounced of late, and has returned to the hypothesis of Fresnel. M. Cauchy supposed too, in the above paper, that the ethereal density is the same in different media; but he has found cause to abandon this hypothesis also. See his notes addressed to M. Libri, in the Comptes rendus des Séances de l'Académie des Sciences, Séance du 4 Avril, 1836, where he gives the reasons for his present opinions. He says, "Ainsi Fresnel a eu raison de dire, non-seulement que les vibrations des molécules éthérées sont généralement comprises dans les plans des ondes, mais encore que les plans de polarisation sont perpendiculaires aux directions des vitesses ou des déplacements moléculaires. J'arrive au reste à cette dernière conclusion d'une autre manière, en établissant les lois de la réflexion et de la réfraction à l'aide d'une nouvelle méthode qui sera développée dans mon mémoire......[cette méthode] ne m'oblige plus à supposer, comme je l'avais fait dans un article du *Bulletin des Sciences*, que la densité de l'éther est la même dans tous les milieux. Mes nouvelles recherches donnent lieu de croire que cette densité varie en général, quand on passe d'un milieu à un autre." More lately, in his Nouveaux Exercices de Mathématiques, 7^e Livraison, M. Cauchy states positively that his principles do not permit him to adopt the hypothesis that the density of the ether is the same in all media. He also gives the differential equations which, as he has found by his new method, ought to subsist at the separating surface of two media, and from which he has obtained the formulæ of Fresnel for ordinary reflexion. But these equations do not include the laws of crystalline reflexion.

Now we are to observe, that when common light is polarised by reflexion at the surface of a doubly refracting crystal, the plane of polarisation does not, in general, coincide with the plane of reflexion, as in the case of ordinary media, but is inclined to it at a certain angle, which may be called the *deviation;* and it was by equating two values of the deviation that I obtained the formula above mentioned for the polarising angle. This formula, as we have seen, was correct; but it happened, singularly enough, that the expressions for the deviation, which

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* Phil. Trans. 1819, p. 150.

† I had, besides, an objection to the law of *vis viva*, on the ground that it would give an equation of the second degree; and I wished to have all my equations linear, lest, in the seemingly complicated question of crystalline reflexion, they should give two answers when the nature of the question required but one. This has actually happened, since the present paper was read, in applying my hypotheses to the case of *internal* reflexion at the second surface of a uniaxal crystal. Supposing an ordinary ray to emerge after double reflexion, and putting θ for the angle which the emergent transversal makes with the plane of incidence, I found, for determining θ, an equation of the form

$$A+B\tan\theta+C\tan^2\theta=0,$$

wherein A is very small, but does not vanish; so that the equation gives two roots, one very small, the other about the proper value. It is clear, therefore, that there is a want of adjustment somewhere; but I am now inclined to think that the fault is not in the principle of *vis viva.* Possibly our laws of the propagation of light in doubly refracting media are not quite accurate. Whatever supplementary law shall be found to remedy this untoward result, will probably, at the same time, account for the extraordinary phenomena observed by Brewster, in reflexion at the *first* surface, when the crystal is in contact with a medium of nearly equal refractive power.

‡ London and Edinburgh Philosophical Magazine, vol. vii. p. 295.

§ Ibid. vol. viii. p. 103; February, 1836.

were used in obtaining the formula, were erroneous. It is to M. Seebeck that I am obliged for pointing out this curious circumstance. In Poggendorff's Annals,* he gave an abstract of my letter to Sir David Brewster, and compared my results with his own numerous and accurate experiments, both on the polarising angles of Iceland spar and on the angles of deviation. He found that my formula represented the former class of experiments as well as could be wished; but the theoretical values of the deviations did not at all agree with his experimental measures. These measures of the deviation he published on this occasion; and, with their assistance, I traced the error to its source, which was the relation among the pressures. The principle of *vis viva* was therefore introduced, instead of that relation, and the theory became much simpler by the change. I now obtained, for the deviation, a new expression, which agreed with the experiments of M. Seebeck; but the formula for the polarising angle came out the very same as before. This correction was made on the 6th of December, and was published in the Philosophical Magazine† on the first of the present month.

In the interval I have arrived at very elegant geometrical laws, which can be easily remembered, and which embrace the whole theory of crystalline reflexion. In enunciating these, it will be convenient to draw our transversals always through the same origin O, which we shall suppose to be the point of incidence, as this point is common to all the rays, whether incident, reflected, or refracted; and we may imagine wave planes to be drawn through the origin, parallel to the plane of each wave, so that every transversal will lie in its own wave plane. The incident and reflected wave planes will be perpendicular to the incident and reflected rays, but the two refracted wave planes will in general be oblique to their respective rays. In the latter case, a right line drawn through the origin perpendicular to the wave plane, is called the wave normal. It is scarcely necessary to remark, that all the four wave planes intersect the surface of the crystal in the same right line which is perpendicular to the plane of incidence; and that the angles of refraction are the angles which the refracted wave normals make with a perpendicular to that surface. The index of refraction is the ratio of the sine of the angle of incidence to the sine of the angle of refraction, just as in ordinary media; but here it is a variable ratio, and has different values for the same angle of incidence. I have elsewhere* shown how to find the refracted rays and waves when the incident ray is given.

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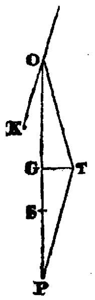
* Annalen der Physik und Chemie, vol. xxxviii. p. 276.

† London and Edinburgh Philosophical Magazine, vol. x. p. 43.

As we suppose the ethereal molecules to vibrate parallel to the transversals, we may take the lengths of the transversals proportional to the magnitudes or amplitudes of the vibrations; these lengths being always measured from the common origin O. Then, in virtue of our fourth hypothesis, the transversals will be compounded and resolved exactly by the same rules as if they were forces acting at the point O.

We must now conceive a wave surface of the crystal, with its centre at O, the point of incidence. As the velocities of rays which traverse the crystal in directions parallel to the radii of its wave surface are represented by those radii, so let a concentric sphere be described with a radius OS, which shall represent, on the same scale, the constant velocity of light in the medium external to the crystal. At any point T on the wave surface apply a tangent plane, on which let fall, from O, a perpendicular OG, meeting the plane in G. On this perpendicular take the length OP from O towards G, so that OP shall be a third proportional to OG and the constant line OS. Then while the point T describes the wave surface, the point P will describe another surface reciprocal† to the wave surface. This other surface may very properly be called the *index surface*,‡ because its radius OP is the refractive index of the ray whose velocity is OT, or rather of the wave TG, which belongs to that ray; for, if we conceive an incident wave, touching the sphere, to be refracted into the wave TG, touching the wave surface in T, the sine of the angle of incidence will be to the sine of the angle of refraction as OS to OG, or as OP to OS; so that, taking the constant OS for unity, the index of refraction will be represented by OP. The wave surface and the index surface will thus be reciprocal to each other,



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* Irish Acad. Trans. vol. xvii. p. 252.

† For the general theory of reciprocal surfaces, see Irish Acad. Trans. vol. xvii. p. 241.

‡ This is the surface which I formerly called (ibid. p. 252) the *surface of refraction;* a name not sufficiently descriptive. Sir W. Hamilton has called it the *surface of wave slowness,* and sometimes the *surface of components.* But the name *index surface* seems to recommend itself, as both short and expressive.

every point T on the one having a point P reciprocally corresponding to it on the other.

It is remarkable that the transversal of the ray OT is perpendicular to the plane OPT; for in the theory of Fresnel, as I formerly proved,* the direction of the vibrations is the right line TG; and as I suppose the transversal to be perpendicular to the vibrations of that theory, and to be, at the same time, in the wave plane, which is perpendicular to OP, it follows that the transversal must be perpendicular to both the right lines TG and OP, and therefore perpendicular to their plane OPT. Therefore conceiving the transversal to be drawn through O at right angles to the plane OPT, the plane of polarisation of the ray OT must needs pass through it. But there is nothing else to fix the position of this last plane. We may make it pass through the ray itself OT, as in ordinary media, or we may draw it through the wave normal OP with Fresnel. Or, instead of drawing it through either of these two sides of the triangle OPT, we may make it parallel to the third side PT. The last is what I should prefer, because the plane so determined possesses important properties. I shall call it, however, the *polar plane,* because the name, plane of polarisation, is a long one; and the signification of the latter may, if any one chooses, be kept distinct, though in an ordinary medium both terms must mean the same thing. The polar plane then of the ray OT is a plane passing through its transversal and parallel to the right line PT; so that if OK be drawn parallel to PT, the polar plane will pass through OK. In general, to find the transversal and the polar plane of any ray, we take the point where the ray meets *its own* nappe of the wave surface, and join it with the corresponding point on the index surface, drawing a plane through the origin and the joining line. Then a right line perpendicular to this plane at the origin will be the transversal, and a plane drawn through the transversal parallel to the joining line will be the polar plane.

Now let a polarised ray be incident at O upon the crystal. It will in general be divided into two rays. But each of these rays in turn may be made to disappear by polarising the incident ray in a certain plane. Let us suppose then that there is only one refracted ray OT. In what direction must the incident ray be polarised, or, in other words, what must be the position of its transversal, in order that this may be the case? and what will be the corresponding transversal of the reflected ray? The answer is simple—*both transversals will lie in the polar plane of the refracted ray.* Let us pursue this remark a little.

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* Ibid. vol. xvi. p. 76.

The refracted ray OT being given, we can find its polar plane, and thence the intersections of this plane with the incident and reflected wave planes. These intersections will be the positions of the incident and reflected transversals when OT is the sole refracted ray. The refracted transversal lies also in the polar plane; and this transversal is, by our fourth hypothesis, the diagonal of a parallelogram, whose sides are the other two transversals; which determines the relative lengths of the three transversals, or the relative amplitudes of the vibrations. The intensities of the reflected and incident rays are, of course, proportional to the squares of their transversals. When the ray OT disappears, we must take the polar plane of the other ray, and proceed as before.

Thus there are, in the incident wave plane, two transversal directions which give only a single refracted ray. These, as well as the corresponding ones in the reflected wave plane, may be called *uniradial* transversals. They are the intersections of the two refracted polar planes with the incident and reflected wave planes.

When the incident transversal does not coincide with either of the uniradial directions, it is to be resolved parallel to them, and then each component transversal will supply a refracted ray, according to the foregoing rules. The reflected transversals, arising from the component incident ones, are to be found separately by the same rules, and then to be compounded.

In ordinary reflexion, if the incident transversal be in the plane of incidence, or perpendicular to it, the reflected transversal will be so likewise. But this does not hold in crystalline reflexion. The general method just given will, however, enable us to determine the positions and magnitudes of the reflected transversals in these two remarkable cases; and then, if we choose, we can reduce any other case to these two, by resolving the incident transversal in directions parallel and perpendicular to the plane of incidence.

If we conceive a pair of incident transversals, at right angles to each other, to revolve about the origin, it is evident that there will be a position in which the reflected transversals corresponding to them will also be at right angles to each other. There is no difficulty in finding this position, and there will be an advantage in using it when common unpolarised light is incident on the crystal. For, the incident transversals being rectangular, we may suppose the light to be equally divided between them, and then the intensities of the corresponding reflected portions can be found by the preceding rules. As the reflected transversals are also rectangular, the sum of these intensities will be the whole intensity of the reflected light, and their difference will be the intensity of the polarised part of it. This part will be polarised in a plane passing through the greater of the two reflected transversals.

Common light will be completely polarised by reflexion when the two uniradial directions in the reflected wave plane coincide with each other; that is, when this plane and the two refracted polar planes have a common intersection. For then, if the incident light be polarised, it is manifest that the reflected transversal will lie in that intersection, whatever be the position of the incident transversal; and therefore if common light be incident, with its transversals in every possible direction, the reflected transversals will have but one direction. Thus the reflected light will be completely polarised in a plane passing through the above intersection.

Hence, as the reflected ray is perpendicular to its wave plane, it follows that, *at the polarising angle of a crystal, the reflected ray is perpendicular to the intersection of the polar planes of the two refracted rays.* The reflected transversal, as we have seen, is this very intersection. This transversal is inclined, in general, to the plane of incidence, and we have had occasion to speak of its inclination under the name of the *deviation.* If we now suppose the double refraction to diminish until it disappears, the intersection of the polar planes will at last coincide* with the refracted ray. There will then be no deviation, and the reflected and refracted rays will be at right angles to each other, agreeably to the law of Brewster, which prevails at the polarising angle of an ordinary medium.

There is a case in which the construction that we have given for determining the polar plane of a ray becomes useless. It is when the ray OT is a normal to the wave surface; for then OP coincides with OT, and we cannot fix the transversal by our construction. But it is precisely in such a case that the polar plane is most easily ascertained, for it is then nothing more than the plane of polarisation of the common theory. For example, if we take the ordinary ray of a uniaxal crystal, its polar plane will pass through the ray itself and the axis of the crystal. Of course in an ordinary medium the polar plane and the plane of polarisation are synonymous.

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* For the polar planes will become two planes of polarisation at right angles to each other.

It may not be amiss to apply our general rules to the case of ordinary reflexion and refraction. Suppose then a polarised ray to fall on the surface of an ordinary medium. Draw a plane through the incident transversal and the refracted ray; this will be the plane of polarisation of the refracted ray, and it will intersect the reflected wave plane in the reflected transversal. The refracted transversal will be the diagonal of a parallelogram, whose sides are the other two transversals; hence we have the relative lengths of the transversals, and thus every thing is determined.*

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* This construction was mentioned at the meeting of the British Association in Dublin.—See the Reports of the Association, or London and Edinburgh Phil. Mag. vol. vii. p. 295. The following is an extract from the paper which I read at that meeting:—

"The formulæ given by Fresnel for the same purpose will be found to agree exactly with this rule, in determining the positions of the planes of polarisation; and his expression for the amplitude of the reflected vibration is also in accordance with our construction. But the coincidence does not hold with regard to the *amplitude* of the refracted vibration, though the *vis viva* of the refracted ray is the same in both theories.

"Now it is very remarkable that if we alter the hypotheses of Fresnel where they are at variance with the preceding principles, we shall, from his own equations of condition, deduce formulæ agreeing in every respect, even as to the amplitude of the refracted wave, with the construction which we have accounted for in a different way, (i. e. by using the relation among the pressures instead of the law of *vis viva.*) The requisite alterations are two in number. First, the vibrations are to be supposed parallel to the plane of polarisation, and not perpendicular to it, as Fresnel conceived; and secondly, the density of the ether is to be considered the same in both media, from which it follows, that the corresponding ethereal masses, imagined by Fresnel, are to each other as the sine of twice the angle of incidence to the sine of twice the angle of refraction. Substituting in Fresnel's equations of condition this value of the ratio of the masses, we obtain the formulæ which I am inclined to regard as correct."

The equations spoken of in this extract, are those which arise from the principle of *vis viva,* and from the equivalence of vibrations parallel to the separating surface of the two media. But it is worth while to observe, that when the vibrations are all in the same direction, that is, when the light is polarised perpendicular to the plane of incidence, the very same formulæ will come out from Young's remarkable analogy of the two elastic balls, one of which impinges directly on the other

The reason of this construction will be evident, if we consider that, in an ordinary medium, the polar plane is the same as the plane of polarisation; and

supposed previously at rest, the masses of the balls being to each other in the ratio of the ethereal masses mentioned above. And, perhaps, this consideration affords the simplest possible explanation of Brewster's law relative to the polarising angle; for, as there is no reflected motion when the balls are equal, the whole velocity of impact being communicated to the ball that was at first quiescent, so there is no reflected vibration when the ethereal masses are equal; that is, when the sine of twice the angle of incidence is equal to the sine of twice the angle of refraction, or when the angles of incidence and refraction are together equal to a right angle. The whole of the incident vibration then passes into the refracted ray. In general, if ι_1, ι_2 denote the angles of incidence and refraction, the masses of the imaginary balls will be as $\sin 2\iota_1$ to $\sin 2\iota_2$; and, if the velocity of the original impact be taken for unity, the common theory of the collision of elastic bodies will give

$$\frac{\sin 2\iota_1 - \sin 2\iota_2}{\sin 2\iota_1 + \sin 2\iota_2} \text{ or } \frac{\tan(\iota_1 - \iota_2)}{\tan(\iota_1 + \iota_2)}, \qquad \text{(i.)}$$

for the velocity retained by the impinging ball after the impact; and

$$\frac{2\sin 2\iota_1}{\sin 2\iota_1 + \sin 2\iota_2} \text{ or } \frac{\sin 2\iota_1}{\sin(\iota_1 + \iota_2)\cos(\iota_1 - \iota_2)}, \qquad \text{(ii.)}$$

for the velocity communicated to the other ball. These expressions, (i.) and (ii.), are the same as the values of τ_3 and τ_2, which we should deduce from equations (1) and (2), on the next page, by supposing τ_1 to be unity, and the angles θ_1, θ_2, θ_3 to be right angles. The general construction given in the text will lead to the same results, if we find from it the limiting ratios of the transversals, on the supposition that their directions approach each other indefinitely, and ultimately coincide in a right line perpendicular to the plane of incidence.

When the transversals are all in the plane of incidence, or when the light is polarised in that plane, the incident, the reflected, and the refracted transversals are to each other as $\sin(\iota_1 + \iota_2)$, $\sin(\iota_1 - \iota_2)$, and $\sin 2\iota_1$ respectively; because each transversal is proportional to the sine of the angle between the other two, and, in the present case, the angle between any two transversals is equal to the angle between the corresponding rays. Hence, taking the incident transversal for unity, the reflected transversal is

$$\frac{\sin(\iota_1 - \iota_2)}{\sin(\iota_1 + \iota_2)}; \qquad \text{(iii.)}$$

and the refracted transversal is

$$\frac{\sin 2\iota_1}{\sin(\iota_1 + \iota_2)}. \qquad \text{(iv.)}$$

It has been already observed, that our theory differs from that of Fresnel with regard to the magnitude of the refracted transversals. The expressions (ii.) and (iv.) must, in fact, be multiplied that, when there is only one refracted ray, the three transversals lie in the polar plane of that ray, according to the general remark with which we set out. We now proceed to show that the theorem asserted in this remark is a consequence of our hypotheses, and we shall afterwards deduce a few results which may be readily compared with experiments.

Let us suppose then that the direction of the incident transversal is such that there is only one refracted ray. It is evident that, in this case, the three transversals must lie in the same plane, since, by the fourth hypothesis, the refracted vibration is the resultant of the other two vibrations; and, therefore, we have only to prove that the plane of the transversals is the same as the polar plane of the refracted ray. Let τ_1, τ_2, τ_3 be the respective lengths of the incident, refracted, and reflected transversals; let θ_1, θ_2, θ_3 be the angles which they make with the plane of incidence, the angle θ_2 being known from the theory of Fresnel; put ι_1, ι_2, ι_3 for the angles made by the respective wave planes with the surface of the crystal, and m_1, m_2, m_3 for the relative quantities of ether set in motion by each wave. Then our hypotheses will give us the four following equations :—

$$m_1\tau_1^2 = m_2\tau_2^2 + m_3\tau_3^2, \qquad (1)$$

$$\tau_1\sin\theta_1 + \tau_3\sin\theta_3 = \tau_2\sin\theta_2, \qquad (2)$$

$$\tau_1\cos\theta_1\cos\iota_1 + \tau_3\cos\theta_3\cos\iota_3 = \tau_2\cos\theta_2\cos\iota_2, \qquad (3)$$

$$\tau_1\cos\theta_1\sin\iota_1 + \tau_3\cos\theta_3\sin\iota_3 = \tau_2\cos\theta_2\sin\iota_2. \qquad (4)$$

The first equation is manifestly the translation of the law of the preservation of *vis viva;* the other three are obtained from the principle of equivalent vibra-

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each by $\frac{\sin\iota_2}{\sin\iota_1}$, in order to produce the corresponding expressions which result from Fresnel's hypotheses. But the two theories also differ as to the relative directions of the incident and reflected transversals. For, supposing the light to fall upon the denser medium, or ι_1 to be greater than ι_2, our construction indicates that these transversals, when the angle of incidence is small, point in the same direction; whereas Fresnel concludes the contrary to be the case. However, the disagreement in this respect ceases as we approach the limiting incidence of 90°; for then, according to both theories, the incident and reflected transversals point in opposite directions. This last conclusion is conformable to the inference which Professor Lloyd has drawn from his experiments on the interference of direct light with light reflected at a very oblique incidence.—See Irish Acad. Trans. vol. xvii. p. 176.

tions, by resolving the vibrations, or transversals, in three rectangular directions. In the second equation, the transversals are resolved perpendicular to the plane of incidence; in the fourth, perpendicular to the surface of the crystal; and in the third equation they are resolved parallel to the intersection of these two planes. When the angles θ_1, θ_2, θ_3 begin, the transversals are in the plane of incidence in such a relative position, that if they were turned round together in that plane through a right angle, they would point each in the direction of its own wave's progress. These angles increase on the same side of the plane of incidence, and range through the whole circumference. The angles ι_1, ι_2, ι_3 are those of incidence, refraction, and reflexion; but, for the sake of symmetry, they are taken to be the angles which the wave normals, drawn from the origin in the direction of each wave's motion, make with the perpendicular to the surface, this perpendicular being directed towards the interior of the crystal. Thus it happens that ι_3 is the supplement of ι_1. Attending to this circumstance, equations (3) and (4) give us

$$\left.\begin{aligned}\tau_1\cos\theta_1-\tau_3\cos\theta_3&=\tau_2\cos\theta_2\,\frac{\cos\iota_2}{\cos\iota_1},\\ \tau_1\cos\theta_1+\tau_3\cos\theta_3&=\tau_2\cos\theta_2\,\frac{\sin\iota_2}{\sin\iota_1};\end{aligned}\right\}\quad(5)$$

and by adding and subtracting these, we find

$$\left.\begin{aligned}\tau_1&=\tau_2\frac{\cos\theta_2}{\cos\theta_1}\,\frac{\sin(\iota_1+\iota_2)}{\sin2\iota_1},\\ \tau_3&=\tau_2\frac{\cos\theta_2}{\cos\theta_3}\,\frac{\sin(\iota_2-\iota_1)}{\sin2\iota_1};\end{aligned}\right\}\quad(6)$$

which values if we substitute in equations (1) and (2), observing that $m_3=m_1$, as is evident, we shall get

$$\frac{\sin^2(\iota_1+\iota_2)}{\cos^2\theta_1}-\frac{\sin^2(\iota_1-\iota_2)}{\cos^2\theta_3}=\frac{m_2}{m_1}\,\frac{\sin^2 2\iota_1}{\cos^2\theta_2},\quad(7)$$

$$\sin(\iota_1+\iota_2)\tan\theta_1-\sin(\iota_1-\iota_2)\tan\theta_3=\sin2\iota_1\tan\theta_2.\quad(8)$$

Subtracting from (7) the identity

$$\sin^2(\iota_1+\iota_2)-\sin^2(\iota_1-\iota_2)=\sin2\iota_1\sin2\iota_2,$$

there remains

$$\sin^2(\iota_1+\iota_2)\tan^2\theta_1-\sin^2(\iota_1-\iota_2)\tan^2\theta_3=\frac{\sin^2 2\iota_1}{\cos^2\theta_2}\left(\frac{m_2}{m_1}-\frac{\sin 2\iota_2}{\sin 2\iota_1}\cos^2\theta_2\right); \qquad (9)$$

and this, by making

$$\frac{m_2}{m_1}=\frac{\sin 2\iota_2+2h\sin^2\theta_2}{\sin 2\iota_1}, \qquad (10)$$

becomes

$$\sin^2(\iota_1+\iota_2)\tan^2\theta_1-\sin^2(\iota_1-\iota_2)\tan^2\theta_3=\sin 2\iota_1(\sin 2\iota_2+2h)\tan^2\theta_2, \qquad (11)$$

which is divisible by equation (8), the quotient being

$$\sin(\iota_1+\iota_2)\tan\theta_1+\sin(\iota_1-\iota_2)\tan\theta_3=(\sin 2\iota_2+2h)\tan\theta_2. \qquad (12)$$

Then, by adding and subtracting equations (8) and (12), we obtain

$$\left.\begin{aligned}\tan\theta_1&=\cos(\iota_1-\iota_2)\tan\theta_2+\frac{h\tan\theta_2}{\sin(\iota_1+\iota_2)},\\ \tan\theta_3&=-\cos(\iota_1+\iota_2)\tan\theta_2+\frac{h\tan\theta_2}{\sin(\iota_1-\iota_2)}.\end{aligned}\right\} \qquad (13)$$

These equations give the positions of the incident and reflected transversals when h is known.

Now let the directions in which the transversals have been resolved in equations (2), (3), (4), be taken for the axes of z, x, y respectively; so that, the origin being at O, the plane of xy may be the plane of incidence, and the axis of x may lie in the surface of the crystal. And, the reflected ray being conceived to lie within the angle made by the positive directions of x and y, let the initial condition that we have assumed for the angles θ_1, θ_2, θ_3 be satisfied by supposing that, when these angles begin, the transversals τ_1, τ_2 lie between the negative directions of x and y, and the tranversal τ_3 between the directions of $+x$ and $-y$. Then if θ_1, θ_2, θ_3 be reckoned towards the positive axis of z, so that each angle may be 90° when the corresponding transversal points in the direction of z positive, the equations of the transversal τ_1 will be

$$\frac{z}{\tan\theta_1}=-\frac{x}{\cos\iota_1}=-\frac{y}{\sin\iota_1}; \qquad (14)$$

and those of τ_3 will be

$$\frac{z}{\tan\theta_3}=\frac{x}{\cos\iota_1}=-\frac{y}{\sin\iota_1}. \tag{15}$$

Let

$$z+\text{A}x+\text{B}y=0, \tag{16}$$

be the equation of a plane passing through the directions of τ_1, τ_2 and τ_3. To determine A and B, let the variables be eliminated from this equation by means of (14) and (15) successively, and we shall get the two equations of condition,

$$\left.\begin{aligned}\tan\theta_1-\text{A}\cos\iota_1-\text{B}\sin\iota_1=0,\\ \tan\theta_3+\text{A}\cos\iota_1-\text{B}\sin\iota_1=0;\end{aligned}\right\} \tag{17}$$

which, by addition and subtraction, give

$$\left.\begin{aligned}\text{B}=\frac{\tan\theta_1+\tan\theta_3}{2\sin\iota_1},\\ \text{A}=\frac{\tan\theta_1-\tan\theta_3}{2\cos\iota_1};\end{aligned}\right\} \tag{18}$$

substituting, in these values, the expressions (13) for $\tan\theta_1$, $\tan\theta_3$, we have

$$\left.\begin{aligned}\text{B}=\tan\theta_2\left(\sin\iota_2+\frac{h\cos\iota_2}{\sin^2\iota_1-\sin^2\iota_2}\right),\\ \text{A}=\tan\theta_2\left(\cos\iota_2-\frac{h\sin\iota_2}{\sin^2\iota_1-\sin^2\iota_2}\right);\end{aligned}\right\} \tag{19}$$

whence, by making

$$\tan\kappa=\frac{h}{\sin^2\iota_1-\sin^2\iota_2}, \tag{20}$$

we find

$$\frac{\text{B}}{\text{A}}=\frac{\tan\iota_2+\tan\kappa}{1-\tan\iota_2\tan\kappa}=\tan(\iota_2+\kappa). \tag{21}$$

But if $z=0$ in (16), we have

$$\text{A}x+\text{B}y=0, \tag{22}$$

for the equation of the right line in which the plane of the transversals intersects the plane of incidence. This right line, lying, like the refracted wave normal, between the directions of $+x$ and $-y$, makes with the direction of $-y$ an angle υ which obviously has $\frac{B}{A}$ for its tangent; and therefore, by (21),

$$\upsilon = \iota_2 + \kappa; \tag{23}$$

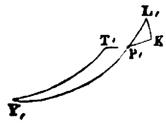
which shows that the intersection of the two planes is inclined to the refracted wave normal at an angle equal to κ.

We must now find the value of h, which depends on the relative ethereal masses put in motion by the incident and refracted waves. Conceiving the incident and refracted rays to be cylindrical pencils, having of course a common section in the plane of xz, which is the surface of the crystal, let each pencil be cut by a pair of planes parallel to its wave plane, and distant a wave's length from each other; then the cylindrical volumes so cut out will represent the corresponding masses, since, by our second hypothesis, the densities are equal. These volumes are to each other in the compound ratio of their altitudes, which are the wave lengths, and of the areas of their bases. The altitudes are evidently as $\sin\iota_1$ to $\sin\iota_2$. The first base is a perpendicular section of the incident pencil; the second base an oblique section of the refracted one, the obliquity being equal to the angle ε at which the wave normal is inclined to the ray. The perpendicular sections are to each other as the cosines of the angles which they make with the common section of the cylinders, or as $\cos\iota_1$ to $\cos\iota_{(2)}$; putting $\iota_{(2)}$ for the angle which the refracted ray makes with the negative direction of y. The second base is greater than the perpendicular section of the refracted pencil in the proportion of unity to $\cos\varepsilon$. Therefore compounding all these ratios, we find

$$\frac{m_2}{m_1} = \frac{\sin\iota_2 \cos\iota_{(2)}}{\sin\iota_1 \cos\iota_1 \cos\varepsilon}. \tag{24}$$

The same result may be otherwise obtained by observing that, in a system of waves, the corresponding masses are proportional to the ordinates y of the points where the rays meet their wave surfaces. By a system of waves, I mean an incident wave with all that are derived from it by reflexion or refraction at the same surface of the crystal, or at parallel surfaces. If at the point where the incident ray intersects its spherical wave surface, we apply a tangent plane intersecting the plane of xz in a right line parallel to z, through which right line other planes are drawn touching the wave surface of the crystal in four points, these tangent planes will be the waves derived from the incident wave which touches the sphere; and the points of contact, including that on the sphere, will be the points where the rays meet the wave surfaces. Then the corresponding masses will be represented by prisms having a common rectangular base in the plane of xz, one side of this rectangle being the distance, on the axis of x, between the origin and the common intersection of the tangent planes; and the triangular face of each prism having the same distance for one side, and a point of contact for the opposite angle. These prisms, as they have a common base, will be proportional to their altitudes, which are the ordinates y of the points of contact. The expression (24) may be easily deduced from this relation.

Let OT, OP, and the negative direction of y meet the surface of the wave sphere (described with the radius OS) in the points $T_{,}$, $P_{,}$, $Y_{,}$; and let the right line, in which the plane of the transversals intersects the plane of incidence, meet the sphere in $L_{,}$. Then the points $Y_{,}$, $P_{,}$, $L_{,}$, being all in the plane of incidence, will be on the same great circle $Y_{,}P_{,}L_{,}$; and drawing the great circles $T_{,}P_{,}$, $Y_{,}T_{,}$, we shall have $Y_{,}P_{,}=\iota_2$, $Y_{,}T_{,}=\iota_{(2)}$, $T_{,}P_{,}=\epsilon$, $Y_{,}L_{,}=\upsilon=\iota_2+\kappa$, by (23); whence $P_{,}L_{,}=\kappa$.



As the transversal τ_2 is perpendicular to the plane OTP, or to the plane of the great circle $T_{,}P_{,}$, the cosine of the spherical angle $T_{,}P_{,}Y_{,}$ is the sine of θ_2; and therefore, from the triangle $T_{,}P_{,}Y_{,}$, we have

$$\cos\iota_{(2)}=\cos\iota_2\cos\epsilon+\sin\iota_2\sin\epsilon\sin\theta_2, \tag{25}$$

which being substituted in (24), gives

$$\frac{m_2}{m_1}=\frac{\sin 2\iota_2+2\sin^2\iota_2\sin\theta_2\tan\epsilon}{\sin 2\iota_1}; \tag{26}$$

and comparing this result with (10), we find

$$h=\frac{\sin^2\iota_2\tan\epsilon}{\sin\theta_2}; \tag{27}$$

whence, and from (20), it follows that

$$\tan\kappa=\frac{\sin^2\iota_2\tan\epsilon}{(\sin^2\iota_1-\sin^2\iota_2)\sin\theta_2}. \tag{28}$$

Draw the great circle $L_{,}K_{,}$ at right angles to $T_{,}P_{,}$, and meeting it in $K_{,}$; then the plane of $L_{,}K_{,}$ will be the plane of the transversals, since the latter plane passes through $L_{,}$, and is perpendicular to $T_{,}P_{,}$. But the tangent of $P_{,}K_{,}$ is equal to the tangent of $P_{,}L_{,}$ multiplied by the cosine of the angle $P_{,}$ or by the sine of θ_2; therefore, denoting $P_{,}K_{,}$ by ε_1, and recollecting that $P_{,}L_{,}=\kappa$, we find, by (28),

$$\frac{\tan\varepsilon_1}{\tan\varepsilon}=\frac{\sin^2\iota_2}{\sin^2\iota_1-\sin^2\iota_2}. \tag{29}$$

Now we have seen that the ratio of OP to OS, or of OS to OG, is the index of refraction; so that $\sin^2\iota_1$ is to $\sin^2\iota_2$ as OP to OG. Therefore, by (29),

$$\frac{\tan\varepsilon_1}{\tan\varepsilon}=\frac{OG}{OP-OG}=\frac{OG}{GP}; \tag{30}$$

but OG is to GP as the tangent of the angle GPT is to the tangent of the angle GOT; and since ε is the angle GOT, it follows that ε_1 is equal to the angle GPT or KOP. Consequently, OK will meet the surface of the sphere in the point $K_{,}$. Thus we have proved our assertion, that, *when there is only one refracted ray, the plane of the transversals is the polar plane of that ray.*

The sign of the quantity h is always the same as that of the cosine of the spherical angle $T_{,}P_{,}Y_{,}$. But to remove all ambiguity respecting signs, we must make a few additional conventions. Supposing, as we have hitherto done, that the refracted light moves from O to T, and conceiving a right line to be drawn from the origin parallel to GT, and directed from G towards T, let the angle ϑ_2, which this right line makes with the plane of incidence, be reckoned, like θ_1, θ_2, from an initial position comprised between the negative directions of x and y; and let ϑ_2, like the angles θ_1, θ_2, θ_3, increase on the side of z positive, and range from 0° to 360°. Then ϑ_2 will always be equal either to the angle $P_{,}$ of the spherical triangle $T_{,}P_{,}Y_{,}$, or to the reentrant angle, which is the difference between $P_{,}$ and 360°. In either case, the cosine of ϑ_2 will be the same, both in magnitude and sign, as the cosine of the angle $T_{,}P_{,}Y_{,}$. Consequently, if, instead of (25), we use the direct trigonometrical formula

$$\cos\iota_{(2)}=\cos\iota_2\cos\varepsilon+\sin\iota_2\sin\varepsilon\cos\vartheta_2, \tag{31}$$

we shall find

$$h=\frac{\sin^2\iota_2\tan\varepsilon\cos\vartheta_2}{\sin^2\theta_2}; \tag{32}$$

showing that the sign of h is always the same as the sign of $\cos\vartheta_2$. Now as θ_2 differs from ϑ_2 by a right angle, we will suppose

$$\theta_2 = \vartheta_2 + 90^\circ, \tag{33}$$

and then we shall have $\sin\theta_2 = \cos\vartheta_2$, algebraically as well as numerically. Thus we see that, by adopting these conventions, the value of h in (27) will have the proper sign. Therefore, substituting this value of h in formulæ (13), we obtain

$$\left.\begin{aligned} \tan\theta_1 &= \cos(\iota_1 - \iota_2)\tan\theta_2 + \frac{\sin^2\iota_2 \tan\varepsilon}{\cos\theta_2 \sin(\iota_1 + \iota_2)}, \\ \tan\theta_3 &= -\cos(\iota_1 + \iota_2)\tan\theta_2 + \frac{\sin^2\iota_2 \tan\varepsilon}{\cos\theta_2 \sin(\iota_1 - \iota_2)}. \end{aligned}\right\} \tag{34}$$

These formulæ give the uniradial directions, or the positions of the incident and reflected transversals, when the sole refracted ray is that with which we have been occupied. The like directions, when the other ray exists alone, will be given by the formulæ

$$\left.\begin{aligned} \tan\theta'_1 &= \cos(\iota_1 - \iota'_2)\tan\theta'_2 + \frac{\sin^2\iota'_2 \tan\varepsilon'}{\cos\theta'_2 \sin(\iota_1 + \iota'_2)}, \\ \tan\theta'_3 &= -\cos(\iota_1 + \iota'_2)\tan\theta'_2 + \frac{\sin^2\iota'_2 \tan\varepsilon'}{\cos\theta'_2 \sin(\iota_1 - \iota'_2)}; \end{aligned}\right\} \tag{35}$$

where all the quantities, except ι_1, which remains the same, are marked with accents, to show that they belong to the second refracted ray.

The uniradial directions having been found by these equations, the relative magnitudes of the uniradial transversals are determined by equations (6). When the incident transversal is not uniradial, it is evident, as we said before, that it may be resolved* in the two uniradial directions; that each component transversal, as if the other component did not exist, will furnish a refracted ray and a partial reflected transversal uniradial in its direction; and that the total (or actual) reflected transversal will be the resultant of the two partial ones.

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* That, if an incident transversal be resolved in any two directions, the reflected and refracted transversals arising from it will be the resultants of those which would arise from each of its components separately, is a principle which appears very evident, insomuch that we can hardly suppose it to be untrue, without doing violence to our physical conceptions. Nevertheless, it is necessary to prove that this principle is not contrary to the law of *vis viva;* for though the *vis viva* may be preserved by each set of components, (as it is when these are uniradial,) yet we cannot *therefore* con-

clude *that it* will be preserved by their resultants. Here then is a test of the consistency of our theory; for we are bound to show that the law of *vis viva* is not infringed by the adoption of the principle in question. Now it is easy to see that, whatever be the two directions in which the incident transversal is resolved, the final results will always be the same; because, taking the component in each of these directions separately, the reflected and refracted transversals belonging to it must be obtained, in the first place, by the help of a resolution performed in the uniradial directions. We need not, therefore, consider any case but that in which the resolution is uniradial throughout.

The incident transversal being denoted by T_1, let T_3 be the reflected transversal determined by the rules given in the text; and let the uniradial components of the former be τ_1, τ'_1, while those of the latter are τ_3, τ'_3. Then will

$$\mathrm{T}_1^2 = \tau_1^2 + \tau'^2_1 + 2\tau_1\tau'_1\cos(\theta_1 - \theta'_1),$$
$$\mathrm{T}_3^2 = \tau_3^2 + \tau'^2_3 + 2\tau_3\tau'_3\cos(\theta_3 - \theta'_3);$$

where the signification of θ_1, θ'_1, θ_3, θ'_3 is the same as in the text. The *vis viva* of one refracted ray is $m_1(\tau_1^2 - \tau_3^2)$, and that of the other is $m_1(\tau'^2_1 - \tau'^2_3)$; therefore the *vis viva* of both refracted rays is

$$m_1(\tau_1^2 + \tau'^2_1 - \tau_3^2 - \tau'^2_3),$$

a quantity which ought to be equal to

$$m_1(\mathrm{T}_1^2 - \mathrm{T}_3^2);$$

and consequently the equation

$$\tau_1\tau'_1\cos(\theta_1 - \theta'_1) = \tau_3\tau'_3\cos(\theta_3 - \theta'_3) \qquad \text{(v.)}$$

ought to be true. This equation, by help of the expressions (6) for τ_1, τ_3, and the like expressions for τ'_1, τ'_3, becomes

$$\sin(\iota_1 + \iota_2)\sin(\iota_1 + \iota'_2)(1 + \tan\theta_1\tan\theta'_1) = \sin(\iota_1 - \iota_2)\sin(\iota_1 - \iota'_2)(1 + \tan\theta_3\tan\theta'_3); \qquad \text{(vi.)}$$

which again, by substituting the values (13) and the other similar values, is changed into

$$\sin(\iota_2 + \iota'_2)\{\cos(\iota_2 - \iota'_2) + \cot\mathrm{an}\,\theta_2\cot\mathrm{an}\,\theta'_2\} + h + h' = 0, \qquad \text{(vii.)}$$

where h' denotes for one refracted ray what h denotes for the other, the value of h being given by formula (27), and that of h' by the same formula with accented letters. The angle of incidence, we may observe, has disappeared from the equation.

If, therefore, the laws of reflexion, which we have endeavoured to establish, are consistent with each other, this last equation must be satisfied by means of the relations which the laws of propagation afford; or rather, the equation must express a property of the wave surface of the crystal, however strange it may be thought that such a property should be derived from the laws of reflexion, laws which would seem, at first sight, to have no connexion at all with the form of the wave surface. Now I have found that the equation (vii.) really does express a *rigorous* property of the biaxal wave surface

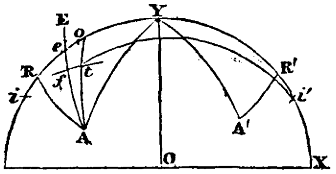
When $\theta_3 = \theta'_3$, the partial reflected transversals will coincide, and their resultant will have a fixed direction, independent of the direction of the incident transversal. The angle of incidence at which this takes place is the polarising angle, and the common value of θ_3 and θ'_3 is the *deviation*. If, at the polarising angle, the partial reflected transversals be equal in magnitude, and opposite in direction, their resultant will vanish, and the reflected ray will disappear. This will happen when the incident transversal is in the plane of the two refracted transversals, and therefore in the intersection of this plane with the incident wave plane; for, when there is no reflected ray, the incident transversal alone must be equivalent to the two refracted transversals.

Since the reflected transversal can be made to vanish at the polarising angle, this angle might be found directly by putting the *vis viva* of the incident ray equal to the sum of the *vires vivæ* of the two refracted rays, and by making the incident transversal the resultant of the two refracted transversals. Resolving the transversals parallel to the axes of coordinates, these conditions would give four equations, from which we could eliminate the two ratios of the three transversals, together with the angle at which the incident transversal is inclined to the plane of incidence. In the equation produced by this elimination, the angle of incidence would be the polarising angle, and the other quantities would be known functions of that angle; whence the angle itself would be known.

It deserves to be remarked, that, at any angle of incidence, if the incident and reflected wave planes be intersected by a plane drawn through the two refracted transversals, the intersections will be corresponding transversal directions; that is to say, if the incident transversal coincide with one intersection, the reflected transversal will coincide with the other. For it is evident, from our fourth hypothesis, that if three of the transversals be in one plane, the fourth transversal must be in the same plane.

of Fresnel; a very curious fact, which not only shows that the laws of reflexion and the laws of propagation are perfectly adapted to each other, but also indicates that both sets of laws have a common source in other and more intimate laws not yet discovered. Indeed the laws of reflexion are not independent even among themselves; for the expressions (iii.) and (iv.) in the note on ordinary reflexion (page 43) have been deduced solely from the principle of equivalent vibrations, and yet they satisfy the law of *vis viva*. Perhaps the next step in physical optics will lead us to those higher and more elementary principles by which the laws of reflexion and the laws of propagation are linked together as parts of the same system.

We come now to apply our theory to the case of uniaxal crystals; and, in doing so, we shall take the crystal to be of the *negative* kind, like Iceland spar, so that the ordinary refraction will be more powerful than the extraordinary. On the sphere described with the centre O and radius OS, let XY be a great circle in the plane of incidence, the radii OX, OY being the positive directions of the coordinate axes of x and y. Suppose the right lines iO and Oi', intersecting the sphere in i and i', to be the incident and reflected rays; let the ordinary refracted ray and the extraordinary wave normal be produced backwards from O to meet the sphere, at the side of the incident light, in the points o and e respectively; let the right line OA, cutting the sphere in A, be the direction of the axis of the crystal; and draw the great circles Ao, Ae, AY. The points i, e, o, i' are all on the circle XY. The point E, where the extraordinary ray OE produced backwards meets the sphere, will be on the circle Ae; and if, as in the figure, the arc Ae be less than a quadrant, the point e will lie between A and E. The polar plane of the ordinary ray is obviously the plane of the circle Ao; but the polar plane of the other ray must be found by a construction. On the arc AeE take the portion ef, so that the point e may lie between the points E and f, and so that the tangent of ef may be to the tangent of Ee as the square of the sine of the arc eY is to the difference between the squares of the sines of iY and eY. Through f draw the great circle ft perpendicular to the circle AeE; and it is manifest from (29) that the plane of ft is the polar plane of the extraordinary ray. On each circumference Ao and ft, the points which are distant 90° from i and i', the distances being measured by arcs of great circles, are the points where the uniradial transversals, prolonged from the centre, intersect the sphere. Let Ao and ft intersect each other in t, and let ti' be an arc of a great circle connecting the point t with the point i'. When the connecting arc ti' is a quadrant, the two uniradial transversals, belonging to the reflected ray, coincide with each other and with the right line Ot; the angle of incidence is then the polarising angle; the plane of ti' is the plane of polarisation of the reflected ray; and the angle ti'Y is the deviation.



To find the equations appropriate to uniaxal crystals, we may suppose formulæ (34) to belong to the ordinary, and formulæ (35) to the extraordinary ray.

Then will $\varepsilon = 0$, and $\varepsilon' =$ the arc Ee. Putting θ and θ' for the spherical angles Aoi and Aei, we shall easily see that $\theta_2 = \theta + 180^\circ$, and $\theta'_2 = \theta' + 90^\circ$, if we conceive the point A and the positive axis of z to be both on the upper side of the plane XOY. And if ω' denote the arc Ae, while b and a respectively express the reciprocals of the principal indices, ordinary and extraordinary, the law of Huyghens, for the double refraction of uniaxal crystals, will give us

$$\tan\varepsilon' = \frac{a^2 - b^2}{s^2} \sin\omega' \cos\omega', \qquad (36)$$

where

$$s^2 = \frac{\sin^2\iota'_2}{\sin^2\iota_1} = b^2 + (a^2 - b^2)\sin^2\omega'. \qquad (37)$$

Observing these relations, we have, from (34),

$$\left.\begin{aligned} \tan\theta_1 &= \cos(\iota_1 - \iota_2)\tan\theta, \\ \tan\theta_3 &= -\cos(\iota_1 + \iota_2)\tan\theta, \end{aligned}\right\} \qquad (38)$$

for the ordinary ray; and from (35) we get

$$\left.\begin{aligned} \tan\theta'_1 &= -\cos(\iota_1 - \iota'_2)\,\mathrm{cotan}\,\theta' - (a^2 - b^2)\frac{\sin\omega'\cos\omega'\sin^2\iota_1}{\sin\theta'\sin(\iota_1 + \iota'_2)}, \\ \tan\theta'_3 &= \cos(\iota_1 + \iota'_2)\,\mathrm{cotan}\,\theta' - (a^2 - b^2)\frac{\sin\omega'\cos\omega'\sin^2\iota_1}{\sin\theta'\sin(\iota_1 - \iota'_2)}, \end{aligned}\right\} \qquad (39)$$

for the extraordinary ray.

The four preceding equations determine the uniradial directions; and the following equation,

$$\cos(\iota_1 + \iota_2)\tan\theta + \cos(\iota_1 + \iota'_2)\,\mathrm{cotan}\,\theta' - (a^2 - b^2)\frac{\sin\omega'\cos\omega'\sin^2\iota_1}{\sin\theta'\sin(\iota_1 - \iota'_2)} = 0, \qquad (40)$$

obtained by putting $\tan\theta_3 = \tan\theta'_3$, is that which determines the polarising angle.

In making use of this last equation to deduce the law of the polarising angles in various positions of the axis of the crystal, we shall confine ourselves to the case in which the reflexion from the crystal takes place *in air*, because the angle $\iota_1 - \iota'_2$ will then be considerable, and the quantities $\cos(\iota_1 + \iota_2)$ and $\cos(\iota_1 + \iota'_2)$ will be small, so that it will be easy to arrive at approximate results. For we shall have, in the first place,

$$\cos(\iota_1+\iota'_2)=\cos(\iota_1+\iota_2)-(\iota'_2-\iota_2) \qquad (41)$$

nearly, since $\iota_1+\iota_2$ will not differ much from a right angle; and because

$$\sin\iota_2=b\sin\iota_1, \quad \sin\iota'_2=s\sin\iota_1, \qquad (42)$$

we shall also have, rigorously,

$$\sin^2\iota'_2-\sin^2\iota_2=(s^2-b^2)\sin^2\iota_1=(a^2-b^2)\sin^2\omega'\sin^2\iota_1, \qquad (43)$$

or

$$\sin(\iota'_2-\iota_2)=(a^2-b^2)\frac{\sin^2\omega'\sin^2\iota_1}{\sin(\iota'_2+\iota_2)}, \qquad (44)$$

which may be written

$$\iota'_2-\iota_2=(a^2-b^2)\frac{\sin^2\omega'\sin^2\iota_1}{\sin 2\iota_2}, \qquad (45)$$

with sufficient accuracy. This value of $\iota'_2-\iota_2$ having been substituted in (41), the resulting expression for $\cos(\iota_1+\iota'_2)$ must be substituted in equation (40), which will then become

$$\cos(\iota_1+\iota_2)(\tan\theta+\operatorname{cotan}\theta)-(a^2-b^2)\sin^2\iota_1\sin\omega\left(\frac{\sin\omega\operatorname{cotan}\theta}{\sin 2\iota_2}+\frac{\cos\omega}{\sin\theta\cos 2\iota_2}\right)=0, \qquad (46)$$

if, denoting the arc A*o* by ω, we confound ω' with ω, θ' with θ, and write $\cos 2\iota_2$ instead of $\sin(\iota_1-\iota'_2)$. Multiplying all the terms of (46) by $\sin\theta\cos\theta$, we find

$$\cos(\iota_1+\iota_2)=(a^2-b^2)\sin^2\iota_1\sin\omega\cos\theta\left(\frac{\sin\omega\cos\theta}{\sin 2\iota_2}+\frac{\cos\omega}{\cos 2\iota_2}\right). \qquad (47)$$

From A draw the arc AR meeting the arc *i*Y at right angles in the point R, and put RY$=p$, AR$=q$. Then by means of the values

$$\left.\begin{aligned}\cos\omega&=\cos q\cos(p-\iota_2),\\ \sin\omega\cos\theta&=\cos q\sin(p-\iota_2),\end{aligned}\right\} \qquad (48)$$

afforded by the right angled triangle AR*o*, the equation (47) will take the form

$$\cos(\iota_1+\iota_2)=\frac{(a^2-b^2)\sin^2\iota_1}{\sin 2\iota_2\cos 2\iota_2}\cos^2 q\sin(p-\iota_2)\sin(p+\iota_2), \qquad (49)$$

or

$$\cos(\iota_1+\iota_2)=\kappa\cos^2 q(\sin^2 p-\sin^2\iota_2), \qquad (50)$$

where

$$\kappa=\frac{(a^2-b^2)(1+b^2)}{2b(1-b^2)};\tag{51}$$

this value of κ being found by assuming $\tan\iota_2=\cot\text{an}\,\iota_1=b$, which is accurate enough for the purpose.

Thus we have obtained $\iota_1+\iota_2$, or the sum of the polarising angle and the angle of ordinary refraction. The former angle itself may be inferred from formula (50) by help of the relation $\sin\iota_2=b\sin\iota_1$. In this way, if we use ϖ_1 instead of ι_1 to distinguish the polarising angle from other angles of incidence, and if we put

$$k=\frac{\kappa}{1+b^2}=\frac{a^2-b^2}{2b(1-b^2)},\tag{52}$$

we shall find

$$\varpi_1=\varpi-k\cos^2q(\sin^2p-\sin^2\iota_2),\tag{53}$$

in which ϖ is the angle whose cotangent is equal to b; in other words, ϖ is the polarising angle of an ordinary medium whose refractive index is equal to the ordinary index of the crystal.

This result accounts for a remarkable fact observed by Sir David Brewster, who, in the year 1819, led the way in the experimental investigation of the laws of crystalline reflexion. He found that the polarising angle remains the same when the crystal is turned round through 180°, though one of the angles of refraction is changed, and though the situation of the refracted rays, with respect to the axis of the crystal, becomes quite different from what it was. This circumstance, which surprised me when I first met with it, is an immediate consequence of formula (53); for the effect of a semi-revolution of the crystal is to change the signs of p and q; but the nature of the formula is such that these changes of sign do not alter the value of ϖ_1. Neither is that value altered by turning the crystal until the azimuth, as the spherical angle AYi is usually called, is changed into its supplement; for then the sign of p alone is affected.

Another remark, made by the same distinguished observer, is also a consequence of formula (53). From his experiments it appears that, on a given surface of the crystal, the polarising angle differs from a constant angle by a quantity proportional to the square of the sine of the azimuth AYi. Now calling this azimuth α, and putting λ for the acute angle at which the axis of the crystal is inclined to its surface, so that λ may be the complement of the arc AY, we have

$$\sin q = \cos\lambda \sin\alpha, \qquad \tan p = \operatorname{cotan}\lambda \cos\alpha; \tag{54}$$

and by making these substitutions in formula (53), after having changed $\sin\iota_2$ into $\cos\varpi$, that formula becomes

$$\varpi_1 = \varpi - k(\sin^2\varpi - \sin^2\lambda) + k\sin^2\varpi\cos^2\lambda\sin^2\alpha, \tag{55}$$

which agrees with the remark of Brewster.

The deviation θ_3 or θ'_3 is found from the second of equations (38), by putting $\frac{\tan q}{\sin(p-\iota_2)}$ for $\tan\theta$, and by substituting for $\cos(\iota_1+\iota_2)$ the value (49) or (50) which it has at the polarising angle. The result is

$$\theta_3 = \theta'_3 = -\frac{\mathrm{K}}{2}\sin 2q \sin(p+\iota_2), \tag{56}$$

since the small arc θ_3 may be taken for its tangent. This result is easily transformed into

$$\theta_3 = \theta'_3 = -\mathrm{K}\sin q \cos\phi, \tag{57}$$

where ϕ denotes the arc A*i*, or the angle which the incident ray makes with the axis of the crystal; and this last expression is equivalent to the following,

$$\theta_3 = \theta'_3 = -\mathrm{K}\cos\lambda \sin\alpha(\sin\lambda\cos\varpi + \cos\lambda\sin\varpi\cos\alpha), \tag{58}$$

which gives the deviation in terms of λ and α.

As an example of the application of our formulæ, we shall make some computations relative to Iceland spar. According to M. Rudberg, the ordinary index of that crystal, for a ray situated in the brightest part of the spectrum, at the boundary of the orange and yellow, is 1.66; and the least extraordinary index for the same ray is 1.487. Dividing unity by each of these numbers, we get $a = .6725$, $b = .6024$; whence $\varpi = 58°\ 56'$; $k = .1164 = 6°\ 40'$; $\mathrm{K} = .1587 = 9°\ 5'$. Having thus determined the constants, we can readily calculate the polarising angle and the deviation, for any given values of λ and α.

First, let us see how the polarising angle varies on different faces of the crystal.

1. When $\lambda = 90°$, the face of the crystal is perpendicular to its axis, and ϖ_1 is independent of α. In this case, the formula (55) gives

$$\varpi_1 = \varpi + k\cos^2\varpi = 60^\circ\ 42',$$

which is the maximum value of the polarising angle.

2. When $\lambda = 0$, the axis lies in the face of the crystal, and formula (55) becomes

$$\varpi_1 = \varpi - k\sin^2\varpi\cos^2\alpha,$$

showing that $\varpi_1 = \varpi$, when α is either 90° or 270°. But when α is 0 or 180°, we have

$$\varpi_1 = \varpi - k\sin^2\varpi = 54^\circ\ 2',$$

which is the minimum value of the polarising angle.

3. For the natural fracture-faces of the crystal the value of λ is 45° 23′. Hence, when $\alpha = 0$ or 180°,

$$\varpi_1 = \varpi - k(\sin^2\varpi - \sin^2\lambda) = 57^\circ\ 26';$$

and when $\alpha = 90^\circ$ or 270°,

$$\varpi_1 = \varpi + k\cos^2\varpi\sin^2\lambda = 59^\circ\ 50'.$$

These values of the polarising angles agree very well with the experiments of Sir David Brewster, and still better with those of M. Seebeck.

If we wish to know in what azimuths ϖ_1 is equal to ϖ, on a given surface of the crystal, it is obvious from (55) that we must make

$$\sin^2\varpi - \sin^2\lambda = \sin^2\varpi\cos^2\lambda\sin^2\alpha,$$

whence we have, simply,

$$\cos\alpha = \pm\frac{\tan\lambda}{\tan\varpi}, \qquad (59)$$

which shows that the thing is impossible when λ is greater than ϖ; and that, when λ is less than ϖ, there are four such azimuths; as indeed there are, generally speaking, four values of α corresponding to any other particular value of the polarising angle. If α' be the least of these azimuths, the others will be $180^\circ - \alpha'$, $180^\circ + \alpha'$, and $360^\circ - \alpha'$. On a natural face of the crystal, the value of α', answering to the supposition $\varpi_1 = \varpi$, is found to be 52° 22′.

Next, let us trace the changes which the deviation undergoes in some remarkable cases.

1. When the face of the crystal is perpendicular to its axis, there is evidently no deviation.

2. When the axis lies in the face of the crystal, the deviation vanishes in the azimuths 0, 90°, 180°, 270°. In the intermediate azimuths, differing 45° from each of these, the deviation is a maximum; for if we put $\lambda = 0$ in formula (55) the result will be

$$\theta_3 = -\frac{\mathrm{K}}{2} \sin\varpi \sin 2\alpha;$$

and this quantity (neglecting its sign) is a maximum when $\sin 2\alpha = \pm 1$. The coefficient of $\sin 2\alpha$ is equal to 3° 54′, which is consequently the greatest value of the deviation. According to the experiments of M. Seebeck, the value is 3° 57′.

3. On the fracture-faces of the crystal, the deviation vanishes in the azimuths 0 and 180°, as also in two other azimuths for which

$$\cos\alpha = -\frac{\tan\lambda}{\tan\varpi},$$

and in which therefore ϖ_1 is equal to ϖ. In the azimuth 45° the deviation is $-3^\circ\ 35'$; in the azimuth 90° it is $-2^\circ\ 32'$; and in the azimuth 127° 38′ it vanishes; after which it attains a small maximum with a positive sign, and vanishes again in azimuth 180°. The calculated values of the deviation agree pretty well with the values observed by M. Seebeck.

The sign of the deviation shows at what side of the plane of incidence the plane of polarisation lies. But the position of the latter plane is best indicated by that of the transversal of the reflected ray. If this transversal and the axis of the crystal be produced from the origin, towards the same side of the plane of xz, until they intersect the sphere in the points t and A respectively, these points will be on the same side of the great circle XY when the deviation and the sine of the azimuth have unlike algebraic signs; and they will be on opposite sides of that circle when those quantities have like signs. Therefore if the crystal be supposed to revolve in its own plane, beginning at the azimuth 0, the points t and A will lie on the same side of XY until A reaches the position A′, where the angle A′Yi is equal to 127° 38′; the point t will then pass over to the side opposite A, at which side it will remain until A arrives at the azimuth 232° 22′. Thenceforward, to the end of the revolution, both points will be found on the same side of the circle XY.

We have seen that the deviation always vanishes when the axis of the crystal lies in the plane of incidence. The reason is, because the crystal is then symmetrical on opposite sides of that plane. In this case the problem of reflexion offers peculiar facilities for solution, since the uniradial directions are obviously parallel and perpendicular to the plane of incidence. Let us therefore consider the case at length.

1. In the first place, when the only refracted ray is the ordinary one, the three transversals are in the plane of incidence, and the transversal of each ray is proportional to the sine of the angle between the other two rays. Hence the proportions are

$$\frac{\tau_1}{\sin(\iota_1+\iota_2)} = \frac{\tau_2}{\sin 2\iota_1} = \frac{\tau_3}{\sin(\iota_1-\iota_2)}, \qquad (60)$$

the same as in ordinary media.

2. In the second place, when the sole refracted ray is the extraordinary one, the three transversals are perpendicular to the plane of incidence; and, if we use accents to mark the quantities connected with this ray, we have the equations

$$\left.\begin{aligned} \tau'_1+\tau'_3&=\tau'_2, \\ m_1\tau'^2_1&=m'_2\tau'^2_2+m_1\tau'^2_3, \end{aligned}\right\} \qquad (61)$$

which give the proportions

$$\frac{\tau'_1}{m_1+m'_2} = \frac{\tau'_2}{2m_1} = \frac{\tau'_3}{m_1-m'_2}, \qquad (62)$$

wherein

$$\frac{m'_2}{m_1} = \frac{\sin 2\iota'_2 \pm 2\sin^2\iota'_2 \tan\epsilon'}{\sin 2\iota_1}, \qquad (63)$$

by (26); the upper or lower sign being taken, in the numerator of (63), according as the refracted ray or its wave normal makes the smaller angle with a perpendicular to the face of the crystal.

To find the polarising angle, we have only to make $m_1=m'_2$, for then τ'_3 will vanish by (62); and therefore, if common light be incident, the whole reflected pencil will be polarised in the plane of incidence. Supposing the crystal to be a negative one, let us conceive the refracted ray to lie within the acute angle made by the axis of the crystal with a perpendicular to its surface. We shall then have to take the positive sign in the numerator of (63), and the polarising angle will be given by the condition

$$\sin 2\iota_1 = \sin 2\iota'_2 + 2\sin^2\iota'_2 \tan \varepsilon'. \qquad (64)$$

But from (36) we have, in general,

$$\sin^2\iota'_2 \tan \varepsilon' = (a^2 - b^2)\sin \omega' \cos \omega' \sin^2\iota_1, \qquad (65)$$

and in the present instance it is evident that

$$\omega' = 90^\circ - \lambda - \iota'_2,$$

where λ denotes, as before, the angle which the axis of the crystal makes with its surface. Substituting these values in (64), and multiplying all the terms by $\tan \iota'_2$, we get

$$\sin^2\iota'_2 = \sin \iota_1 \cos \iota_1 \tan \iota'_2 - (a^2 - b^2)\sin(\lambda + \iota'_2)\cos(\lambda + \iota'_2)\tan \iota'_2 \sin^2\iota_1.$$

Again, from (37) we have

$$\sin^2\iota'_2 = b^2 \sin^2\iota_1 + (a^2 - b^2)\cos^2(\lambda + \iota'_2)\sin^2\iota_1; \qquad (66)$$

and by equating these two expressions for $\sin^2\iota'_2$, we find

$$\tan \iota'_2 = \frac{a^2 \cos^2\lambda + b^2 \sin^2\lambda}{\operatorname{cotan} \iota_1 + (a^2 - b^2)\sin \lambda \cos \lambda}. \qquad (67)$$

Then if this value of $\tan \iota'_2$ be substituted in equation (66), after all its terms have been divided by $\cos^2\iota'_2$, we shall obtain the simple and rigorous formula

$$\sin^2\iota_1 = \frac{1 - a^2 \cos^2\lambda - b^2 \sin^2\lambda}{1 - a^2 b^2} = \sin^2\varpi_1, \qquad (68)$$

for determining the polarising angle ϖ_1, when the axis of the crystal lies in the plane of incidence. It is manifest, from the nature of the formula, that this angle is the same, whether the azimuth is 0 or 180°; that is, whether the light is incident at the right or left side of the perpendicular to the surface of the crystal.

This formula might be deduced more briefly by recollecting what we have already proved, that the corresponding masses m_1 and m'_2 are proportional to the ordinates y of the points where the incident ray and the extraordinary refracted ray meet their respective wave surfaces; whence it follows, that these ordinates must be equal at the polarising angle; and thus the question is reduced at once to a geometrical problem. For as both rays are in the plane of incidence, the axis of x will be intersected in one and the same point by right lines touching the wave surfaces, or their sections, at the extremities of the ordinates. Now the sections in the plane of xy are a circle and ellipse with their common centre at the origin, the radius of the circle being unity, and the semiaxes of the ellipse being a and b, of which b is inclined at the angle λ to the axis of x; and therefore it is required to draw, parallel to the axis of x, a right line intersecting the circle and ellipse, so that if tangents be applied to them at two points of intersection which lie on the same side of the axis of y, these tangents, when produced, may cut each other on the axis of x. The angle which the tangent to the circle makes with the axis of x is then the polarising angle ϖ_1; and the solution of the problem just stated leads directly and easily to the formula (68). From this way of viewing the matter we see the reason why the polarising angle is the same in the azimuths 0 and 180°; for if tangents be applied at the two remaining points where the parallel that we have spoken of intersects the circle and ellipse, it is evident that these tangents also will cut each other on the axis of x; since tangents drawn at the extremities of any chord, either of a circle or an ellipse, intersect the parallel diameter at equal distances from the centre.

Let the reflecting surface of the crystal be in contact with a fluid medium whose index of refraction out of vacuo is represented by N, and let B and A respectively denote the ordinary and the principal extraordinary indices of refraction out of vacuo into the crystal. Then putting $\frac{\text{N}}{\text{A}}$ for a, and $\frac{\text{N}}{\text{B}}$ for b, in the preceding formula, and making

$$\text{L}^2 = \text{A}^2\sin^2\lambda + \text{B}^2\cos^2\lambda,$$

we readily deduce

$$\tan^2\varpi_1 = \frac{\text{A}^2\text{B}^2 - \text{L}^2\text{N}^2}{\text{N}^2(\text{L}^2 - \text{N}^2)}. \qquad (69)$$

Hence we perceive that if $\text{L}^2 = \text{AB}$, that is, if

$$\tan\lambda = \sqrt{\frac{\text{B}}{\text{A}}},$$

(in which case λ will never be much above or below 45°,) the value of ϖ_1 will be always possible; for then we shall have

$$\tan^2\varpi_1 = \frac{\text{AB}}{\text{N}^2}. \qquad (70)$$

But if λ be different from this, and of course L not equal to AB, the value of ϖ_1

may become impossible for certain values of N. For it is clear that if N lie *between* the limits L and $\frac{AB}{L}$, the numerator and denominator of the fraction (69) will have unlike signs, and the tangent of ϖ_1 will be the square root of a negative quantity. In this case, therefore, if common light be incident, it will "refuse to be polarised," as Brewster expresses it; in other words, it will be impossible to find an angle of incidence at which the reflected pencil will cease to contain light polarised perpendicularly to the plane of incidence, or at which the reflected transversal τ'_3 will vanish. With all values of N, except those which are included between the narrow limits L and $\frac{AB}{L}$, the polarising angle is possible. It is zero at the latter limit, and 90° at the former. Outside these limits it changes rapidly at first, until N has passed either of them by a quantity considerable in proportion to the interval between them.

From (68) we find $\varpi_1 = \lambda$, when $a = 1$, or $N = A$; and also $\varpi_1 = 90° - \lambda$, when $b = 1$, or $N = B$. In the latter case it is remarkable that no light is reflected when common light is incident at the angle $90° - \lambda$. For then we have $\tau'_3 = 0$; and, because $\iota_1 = \iota_2$, we have likewise $\tau_3 = 0$. Therefore no light can enter the reflected pencil. But this case deserves that we should consider it more at large, without restricting ourselves to the supposition that the axis of the crystal lies in the plane of incidence.

Assuming then that $N = B$, or that the refractive index of the fluid, which covers the reflecting surface, is equal to the ordinary index of the crystal itself, we may observe that, in this case, every angle of incidence, in every azimuth, has a right to be regarded as a polarising angle. In fact, common light cannot suffer reflexion at the separating surface of the crystal and the fluid, without becoming completely polarised. For if polarised light be incident, and if τ_3 and τ'_3 be the uniradial reflected transversals, respectively belonging to the ordinary and to the extraordinary ray, the former transversal must necessarily vanish, for the same reason that no reflexion can take place at the separating surface of two ordinary media whose refractive indices are equal; and thus the actual reflected transversal will always coincide in direction with τ'_3, whatever be the direction of the incident transversal. Consequently, if common light be incident, the whole reflected pencil will be polarised in a plane passing though τ'_3, and making with the plane of incidence an angle θ'_3 determined by the second of formulæ (39). By putting $\iota_2 = \iota_1$ in that formula, and employing the expression (44), we first obtain

$$\tan\theta_3 = \frac{\cos(\iota_1 + \iota'_2)\cos\theta'\tan\omega' + \sin(\iota_1 + \iota'_2)}{\sin\theta'\tan\omega'} = \frac{\cos(\iota_1 + \iota'_2)\tan(p - \iota'_2) + \sin(\iota_1 + \iota'_2)}{\sin\theta'\tan\omega'};$$

and thence

$$\tan\theta'_3 = \frac{\sin(p + \iota_1)}{\sin\theta'\tan\omega'\cos(p - \iota'_2)} = \frac{\sin(p + \iota_1)\cos\omega'}{\sin q\cos(p - \iota'_2)};$$

and finally,

$$\tan\theta'_3 = \sin(p + \iota_1)\,\mathrm{cotan}\,q; \tag{71}$$

a result which shows that the plane of polarisation of the reflected ray is perpendicular to a plane drawn through the ray itself and the axis of the crystal.

Moreover, we find, from the first of formulæ (39), by proceeding as above,

$$\tan\theta'_1 = -\sin(p - \iota_1)\,\mathrm{cotan}\,q = -\,\mathrm{cotan}\,\theta; \tag{72}$$

and from (38) it is evident that $\theta_1 = \theta$. Therefore all that relates to the case under our consideration may be summed up in the following statement:

When N = B, and the incident light is polarised in a plane passing through the axis, the course of the light is unaltered, and there is neither reflexion nor refraction. When it is polarised in the perpendicular plane, all the light which enters the crystal undergoes extraordinary refraction. Whatever light is reflected, is always polarised in a plane at right angles to that which passes through the reflected ray and the axis of the crystal; and this is true, whether the incident light is polarised or not.

Here, for the present, we must terminate our deductions from the general theory propounded in this paper. Several other questions remain to be discussed, such as the reflexion of common light* at the first surface, and the internal† reflexion at the second surface of a crystal; but these must be reserved for a future communication. It would be easy, indeed, to write down the algebraical

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* The mode of treating the case in which common light is incident, has been pointed out at the bottom of p. 40.

† I have since found that the problem of reflexion at the second surface may be reduced to that of reflexion at the first surface, by means of a very simple rule. Let us suppose the two surfaces of the crystal to be parallel; and let a ray R_1, uniradially polarised, and incident on the first surface, give the ray R_3 by reflexion, and the single ray R_2 by refraction. Let R_2 be the ray which suffers internal reflexion at the second surface, thereby giving the two reflected rays $R_{,,}$, $R'_{,,}$, and the single

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refracted ray $R_{(1)}$ emerging from the crystal in a direction parallel to R_1. Put τ_1, τ_3, τ_2, and $\tau_{\prime\prime}$, $\tau'_{\prime\prime}$, $\tau_{(1)}$ for the transversals of the rays in the order in which they have been named. As the transversal τ_2 is supposed to be given in magnitude, the lengths as well as the directions of τ_1 and τ_3 can be found by the construction in page 40.

Now, the direction of τ_3 being changed, and its magnitude retained, let the ray R_3 be turned directly back, so as to be incident again on the crystal, and to suffer reflexion and refraction at the first surface. Then the two refracted rays which it gives will be parallel to $R_{\prime\prime}$, $R'_{\prime\prime}$, and their transversals will be equal and parallel to $\tau_{\prime\prime}$, $\tau'_{\prime\prime}$. The reflected ray which it gives will coincide with R_1; and the reflected transversal, when compounded with τ_1, will furnish a resultant equal and parallel to the emergent transversal $\tau_{(1)}$.

Thus the constructions, which have been given for the first surface, may be made available for the second surface, and every question relative to crystalline reflexion may be solved geometrically by means of the polar planes.

The foregoing rule was not, properly speaking, deduced from theory. I first formed a clear conception of what the rule ought to be, and then verified it for the simple case of singly refracting media, and finally proved it for doubly refracting crystals. The truth of the rule, in crystals, depends upon the truth of the three following equations:

$$\left.\begin{array}{l}\sin(\iota_{\prime\prime}+\iota'_{\prime\prime})\,\{\cos(\iota_{\prime\prime}-\iota'_{\prime\prime})+\text{cotan}\,\theta_{\prime\prime}\,\text{cotan}\,\theta'_{\prime\prime}\}\;+h_{\prime\prime}+h'_{\prime\prime}=0,\\ \sin(\iota_2-\iota_{\prime\prime})\,\{\cos(\iota_2+\iota_{\prime\prime})-\text{cotan}\,\theta_2\,\text{cotan}\,\theta_{\prime\prime}\}\;+h_2-h_{\prime\prime}=0,\\ \sin(\iota_2-\iota'_{\prime\prime})\,\{\cos(\iota_2+\iota'_{\prime\prime})-\text{cotan}\,\theta_2\,\text{cotan}\,\theta'_{\prime\prime}\}\;+h_2-h'_{\prime\prime}=0,\end{array}\right\}\qquad\text{(viii.)}$$

in which the notation is intelligible without any explanation. The first equation is the same as equation (vii.) already noticed; and the other two differ from it only in appearance, the change in the signs being occasioned by a change in the relative position of the rays.

When the reflexion is total, I suppose we may follow the example which Fresnel has set us in the case of ordinary media. The general algebraic expression for each reflected transversal will then become imaginary; and by putting it under the form

$$T\,(\cos\Phi+\sqrt{-1}\,\sin\Phi),$$

we shall have T for the reflected transversal, and Φ for the change of phase.

From the nature of the rules which we have given for treating the question of reflexion at either surface of the crystal, it follows that the final equation, for determining the position of a transversal, is always linear, though the equation of *vis viva* is of the second degree. This result very strongly confirms the theory; but it shows, at the same time, that the law of the preservation of *vis viva* is not to be regarded as an ultimate principle, but rather as a consequence of some elementary law not yet discovered.

It now appears that the conjectures put forward in the note, p. 36, were hasty, and that there was some mistake in the calculations which gave rise to them. It is scarcely necessary to mention, solutions resulting from our theory; but this we are not content to do, because the expressions are rather complicated, and, when rightly treated, will probably contract themselves into a simpler form. It is the character of all true theories that the more they are studied, the more simple they appear to be. And we may add, that a close examination of such theories always meets with its reward, in the unexpected* consequences which present themselves to view. Nothing can

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that the sheet in which that note is found was printed off before I had obtained the result announced in the subsequent note, p. 52. Various delays occurred while my paper was going through the press; and I took advantage of them to increase its value, by appending notes on some of the questions which I had overlooked or omitted in the first consideration of the subject.

* As an instance of this, it may be mentioned, that the conclusion arrived at in the note, p. 52, was wholly unexpected. And in verifying the equation (vii.), an unexpected and useful theorem was obtained; for it became necessary to find a manageable expression for the tangent of the angle ε which the wave normal makes with the ray. This expression is wanted in applying the formulæ (34) and (35) to biaxal crystals, and therefore I shall make no apology for introducing it here.

Having described a sphere concentric with the wave surface, let the wave normal OP and the two optic axes (which are the nodal diameters of the *index surface*,) be produced from the centre O to meet the sphere in the points $P_{,}$, A, $A_{,}$, respectively, thus marking out the angles of a spherical triangle $P_{,}AA_{,}$. The same wave normal may belong to two different rays; and if we select one of these rays, its transversal must lie in a plane drawn through the wave normal, and bisecting either the internal angle $AP_{,}A_{,}$ of the spherical triangle, or the external supplementary angle. By producing the optic axes in the proper directions, we may always make the above plane (which Fresnel calls the plane of polarisation) bisect the internal angle. Supposing this to have been done for the ray which was selected, put ω and $\omega_{,}$ for the sides $P_{,}A$ and $P_{,}A_{,}$ of the spherical triangle, and ψ for the contained angle $AP_{,}A_{,}$. Let s be the length of the wave normal from the centre O to the point where it intersects the tangent plane applied at the extremity of the ray, that is, applied at the point where the ray meets its own nappe of the wave surface; and let a and c be the greatest and least semiaxes of the ellipsoid which generates the wave surface. Then we shall have

$$\tan\varepsilon = \frac{a^2-c^2}{2s^2}\sin(\omega-\omega_{,})\sin\tfrac{1}{2}\psi. \qquad \text{(ix.)}$$

And it is now manifest that if $\varepsilon_{,}$ be the angle which the other ray makes with the same wave normal, and $s_{,}$ the length of the wave normal intercepted between the centre and the tangent plane at the extremity of this ray, we shall also have

$$\tan\varepsilon_{,} = \frac{a^2-c^2}{2s_{,}^2}\sin(\omega+\omega_{,})\cos\tfrac{1}{2}\psi. \qquad \text{(x.)}$$

If a ray is given in direction it will have two wave normals; and then the angles ε, $\varepsilon_{,}$, which it makes with each normal may be found from the formulæ be simpler than the laws of double refraction, as they were delivered by Fresnel; yet the properties of his wave surface still continue to furnish the geometer with beautiful and curious relations. So we may hope that a little more time, devoted to the laws of reflexion, will not be spent in vain. They promise to supply many other theorems, not undeserving of attention, though perhaps not as simple and comprehensive as those that have already been made known.

If we are asked what reasons can be assigned for the hypotheses on which the preceding theory is founded, we are far from being able to give a satisfactory answer. We are obliged to confess that, with the exception of the law of *vis viva*, the hypotheses are nothing more than fortunate conjectures. These conjectures are very probably right, since they have led to elegant laws which are fully borne out by experiments; but this is all that we can assert respecting them. We cannot attempt to deduce them from first principles; because, in the theory of light, such principles are still to be sought for. It is certain, indeed, that light is produced by undulations, propagated, with transversal vibrations, through a highly elastic ether; but the constitution of this ether, and the laws of its connexion (if it has any connexion) with the particles of bodies, are utterly unknown. The peculiar mechanism of light is a secret that we have not yet been able to penetrate. As a proof of this, we might observe, that some of the simplest and most familiar phenomena have never been explained. Not to mention dispersion, about which so much has been fruitlessly written, we may remark, that the very cause of ordinary refraction, or of the retardation which light undergoes upon entering a transparent medium, is not at all understood. Much less can it be said that double refraction has been rigorously explained; its laws alone have been clearly developed by Fresnel. In short, the whole amount of our knowledge, with regard to the propagation of light, is confined to the *laws* of phenomena; scarcely any approach has been made to a mechanical theory of those laws. And if the

$$\left.\begin{aligned}\tan\varepsilon &= \frac{r^2}{2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin(\omega-\omega_{\prime})\sin\tfrac{1}{2}\psi,\\ \tan\varepsilon_{\prime} &= \frac{r_{\prime}^2}{2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin(\omega+\omega_{\prime})\cos\tfrac{1}{2}\psi,\end{aligned}\right\}\qquad\text{(xi.)}$$

where r and $r_{\prime}$ are the two radii of the wave surface which are in the direction of the ray; the spherical triangle $P_{\prime}AA_{\prime}$, of which the sides and contained angle are expressed by the same letters as before, being now formed by producing the *ray* and the two nodal diameters of the *wave surface*, until they intersect the sphere in the points $P_{\prime}$, A, $A_{\prime}$.

case of uninterrupted propagation through a continuous medium presents such difficulties, it would be useless to think of accounting for the laws which subsist at the confines of two media, where the continuity is broken.

But perhaps something might be done by pursuing a contrary course; by taking those laws for granted, and endeavouring to proceed upwards from them to higher principles. In this point of view, our second law, or hypothesis, is extremely remarkable; for it seems to be opposed, in some degree, to the notion that the ethereal molecules are strongly attracted or repelled by the particles of bodies. However that may be, it would appear that a true theory must be in accordance with this hypothesis; and that any mechanical ideas, which would make the *mean* density of the ether vary from one medium to another, cannot be admitted to represent the real state of things in nature.* It is no objection to the hypothesis in question, to say that it increases the difficulty of accounting for refraction; for, as there is positive evidence in favour of the hypothesis, we ought rather to conclude that the common opinion, which attributes refraction to a change of density in the ether, is altogether erroneous.

In the next place we may remark, that our first hypothesis,† concerning the direction of vibrations in polarised light, will be useful in testing any proposed theory; for as it now seems to be certain that the vibrations are parallel to the plane of polarisation, and not perpendicular to it, as Fresnel supposed, such a direction of the vibrations ought to be a consequence of the theory which we adopt.

The third hypothesis, or the principle of the preservation of *vis viva*, is the most natural that can be imagined, inasmuch as it implies only this, that the incident light is equal to the sum of the reflected and refracted lights. Yet it is probable that even this principle, like the law of *vis viva* in ordinary mechanics, is a result of simpler laws, and will be shown to be so, as soon as the true mechanism of light shall be discovered.

The fourth hypothesis is a very important one, because the whole theory turns upon it; and therefore, in the beginning of this paper, a particular account has been given of the manner in which it was originally suggested. If we wished to give a reason for this hypothesis, we might say, that the motion of a particle of ether, at the common surface of two media, ought to be the same, to whichsoever medium the particle is conceived to belong; and as the incident and reflected vibrations are superposed in one medium, and the refracted vibrations in the other, we might infer that the resultant of the former vibrations ought to be the same, both in length and direction, as the resultant of the latter. At first sight this reasoning appears sufficiently plausible; but it will not bear a close examination. For as the argument is general, it would prove that the principle of the equivalence of vibrations is true for metals,* as well as for crystals, which it certainly is not.

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* Those who maintain that the density of the ether is different in different media, ought to consider the following question:—What function of the three principal indices of a doubly refracting crystal, represents the density of the ether within the crystal?

† This hypothesis properly belongs to the laws of propagation, as it relates only to what passes within a given medium.

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* A few days after this paper was read, I found reason to persuade myself, that, in metals, the vibrations parallel to the surface are equivalent, but not those perpendicular to it; and that, in metals as well as in crystals, the *vis viva* is preserved. This persuasion was founded on a system of formulæ which I had invented for expressing the laws of metallic reflexion and refraction; and which seem to represent very satisfactorily the experiments of Brewster, Phil. Trans. 1830. As metallic and crystalline reflexion are kindred subjects, and will one day be brought under the same theory, however distinct they may now appear, it will not be out of place to insert the formulæ for metals here. These formulæ are not proposed as true, but as likely to be true; and they will be found to express, at least with general correctness, all the circumstances that have hitherto been regarded as anomalies in the action of metals upon light.

I suppose that, for every metal, there are two constants, M and χ, of which the first is a number greater than unity, and the second is an angle included between 0 and 90°. The number M I call the *modulus*, and the angle χ the *characteristic* of the metal. Both M and χ vary with the colour of the light, and the ratio $\frac{\text{M}}{\cos\chi}$ is probably the index of refraction. From Brewster's experiments it appears that M diminishes from the red to the violet; and therefore I should suppose that $\cos\chi$ diminishes in a greater ratio, in order that the index of refraction may increase as in transparent substances.

Put ι_1 for the angle of incidence, and ι_2 for the angle of refraction, so that

$$\frac{\sin\iota_1}{\sin\iota_2} = \frac{\text{M}}{\cos\chi}; \qquad \text{(xii.)}$$

and let μ be a variable determined by the condition

$$\mu = \frac{\cos\iota_1}{\cos\iota_2}. \qquad \text{(xiii.)}$$

These two relations combined will give

$$\frac{1}{\mu^2} = 1 + \left(1 - \frac{\cos^2\chi}{\text{M}^2}\right)\tan^2\iota_1, \qquad \text{(xiv.)}$$

It is not easy to see why the principle should hold in the one case, and not in the other; but it is probably prevented from holding, in the case of metals, by the

which shows that μ is equal to unity at a perpendicular incidence, and that it vanishes at an incidence 90°, decreasing always during the interval.

Now if plane polarised light be incident on the metal, we must distinguish two principal cases, according as the light is polarised in the plane of incidence, or in the perpendicular plane. In the first case, denoting the reflected and refracted transversals by τ_3 and τ_2 respectively, let us put Δ_3 for the change of phase in the reflected ray, and Δ_2 for the change of phase in the refracted ray. Let the same symbols, marked with accents, be used in the second case with similar significations. Then if the incident transversal be taken for unity, we shall have the following formulæ:

1. When the incident transversal is in the plane of incidence,

$$\left.\begin{aligned} \tau_3{}^2 &= \frac{\mathrm{M}^2+\mu^2-2\mathrm{M}\mu\cos\chi}{\mathrm{M}^2+\mu^2+2\mathrm{M}\mu\cos\chi}, \\ \tau_2{}^2 &= \frac{4\mathrm{M}^2\mu^2}{\mathrm{M}^2+\mu^2+2\mathrm{M}\mu\cos\chi}, \\ \tan\Delta_3 = \frac{2\mathrm{M}\mu\sin\chi}{\mathrm{M}^2-\mu^2}, &\qquad \tan\Delta_2 = \frac{\mu\sin\chi}{\mathrm{M}+\mu\cos\chi}. \end{aligned}\right\} \quad \text{(xv.)}$$

2. When the incident transversal is perpendicular to the plane of incidence,

$$\left.\begin{aligned} \tau'_3{}^2 &= \frac{1+\mathrm{M}^2\mu^2-2\mathrm{M}\mu\cos\chi}{1+\mathrm{M}^2\mu^2+2\mathrm{M}\mu\cos\chi}, \\ \tau'_2{}^2 &= \frac{4\mathrm{M}^2\mu^2}{1+\mathrm{M}^2\mu^2+2\mathrm{M}\mu\cos\chi}, \\ \tan\Delta'_3 = \frac{2\mathrm{M}\mu\sin\chi}{\mathrm{M}^2\mu^2-1}, &\qquad \tan\Delta'_2 = \frac{\sin\chi}{\mathrm{M}\mu+\cos\chi}. \end{aligned}\right\} \quad \text{(xvi.)}$$

When $\chi=0$, there is no change of phase, and the formulæ become identical with those given in the note, p. 43. When $\chi=90°$, there is total reflexion at all incidences. The case of pure silver approximates to this. For good speculum metal, χ is about 70°. The value of M ranges from $2\frac{1}{2}$ to 5 in different metals.

When the incident transversal is inclined to the plane of incidence, its components, parallel and perpendicular to that plane, will give two reflected transversals with a difference of phase equal to $\Delta'_3-\Delta_3$. The reflected vibration will then be performed in an ellipse; and the position and magnitude of the axes of the ellipse may be deduced from the preceding formulæ. The consequences of these formulæ are very simple and elegant, but I cannot dwell upon them here. Suffice it to observe, that every angle of incidence has another angle corresponding to it, which I call its *conjugate* angle of incidence; and that the value of $\Delta'_3-\Delta_3$ at one of these angles is the supplement of its value at the other, while the ratio $\frac{\tau'_3}{\tau_3}$ is the same at both angles; whence it follows that, *ceteris paribus*, the elliptic vibrations, reflected at conjugate angles, are similar to each other, and have their homologous axes equally inclined to the plane of incidence, but on opposite sides of it. When same cause, whatever it is, which produces a change of phase in metallic reflexion.

It will be proper to conclude this Essay with a brief sketch of the researches of Sir David Brewster and M. Seebeck, the only other writers who have treated of the subject of crystalline reflexion.

So early as the year 1819, Sir David Brewster published, in the Philosophical Transactions, a paper "On the Action of Crystallized Surfaces upon Light."* In this paper the author details a great variety of experiments on the polarising effects of Iceland spar. He gives the measures of the polarising angles in different azimuths, when the reflexion takes place in air; but he does not notice the accompanying deviations, which were probably too small to attract his attention. In another instance, however, he obtained very large deviations. He conceived the idea of pushing his experiments into an extreme case, by masking, as it were, the ordinary reflecting action of the crystal, and leaving the extraordinary energy at full liberty to display itself. This was done by dropping on the reflecting surface a little oil of cassia, a fluid whose refractive index is nearly equal to the ordinary index of Iceland spar. When common light, incident at 45°, was reflected at the separating surface of the oil and the spar, the reflected pencil was found to be partially, and sometimes completely, polarised in planes variously inclined to the plane of incidence, the inclination going through all magnitudes from 0 to 180°, as the crystal was turned round in azimuth. This general result is no more than what theory would lead us to expect, when the angle of incidence is nearly equal to one of the angles of refraction; but, to institute a minute comparison of theory with experiment, would require troublesome calculations, which I have not had time to make. With the view, however, of showing clearly, from theory, that the range of the deviation is unlimited, I have considered the simple case in which $\mathrm{N} = \mathrm{B}$, or in

$\Delta'_3 - \Delta_3 = 90°$, the conjugate incidences are equal, the ratio $\frac{\tau'_3}{\tau_3}$ is a minimum, and the axes of the elliptic vibration are parallel and perpendicular to the plane of incidence. When $\Delta'_3 = 90°$, or $\mathrm{M}\mu = 1$, the value of τ'_3 is a minimum, and equal to $\tan \frac{1}{2} \chi$.

The foregoing formulæ differ slightly from those which I have given in No. I. of the Proceedings of the Royal Irish Academy. The small quantity χ', which occurs in the latter, has been purposely neglected, as its presence interferes with the simplicity of the expressions.

* Phil. Trans. 1819, p. 145.

which the refractive index of the fluid is *exactly* equal to the ordinary index of the crystal. This case, moreover, is remarkable on its own account; and it might be worth while to try whether it could not be verified by direct experiment. If a fluid could be procured, whose refractive index, for some definite ray of the spectrum, should be equal to the ordinary index of the crystal for the same ray, and if common light, incident at any angle and in any azimuth, were reflected at the confines of the fluid and the crystal, then, supposing the theory to be exact, the definite ray aforesaid would, as we have seen, be completely polarised by reflexion, and the plane of polarisation would always be perpendicular to a plane drawn through the direction of the reflected ray and the axis of the crystal. This experiment would be an elegant test of the theory in its application to these extreme and trying cases; and if it were successful, no doubt could be entertained* as to the rigorous accuracy of the geometrical laws of reflexion.

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* I was at this time in doubt whether the phenomena observed with oil of cassia could be reconciled to theory, and when the note in page 36 was written, I was almost certain that they could not. But I have since, I think, found out the cause of this perplexity. Some of Brewster's experiments were made with *natural* surfaces of Iceland spar; others with surfaces *artificially polished.* I believe (though I have made very few calculations relative to the point) that the former class of experiments will be perfectly explained by the theory; the latter I am certain cannot be so explained, nor ought we to expect that they should. For the process of artificial polishing must necessarily occasion small inequalities, by exposing little elementary rhombs with their faces inclined to the general surface; and the action of these faces may produce the *unsymmetrical* effects which Brewster notices as so extraordinary (Sixth Report of the British Association, Transactions of the Sections, p. 16). If this will not account for such effects, I do not know what will. From an old observation of Brewster (Phil. Trans. 1819) it would appear, that imperfect polish does actually produce a want of symmetry in the phenomena; for when common light was reflected between oil of cassia and a badly polished surface *perpendicular* to the axis, he found that the reflected ray was polarised neither in the plane of incidence, nor perpendicular to it, but 75° out of it. The same surface, when the light was reflected in air, gave the polarising angle more than two degrees below its proper value.

To show that, in other respects, the general character of the phenomena is in accordance with theory, we may observe that, when $N=B$, and $\lambda=0$ or 90°, if common light be incident at 45° in the plane of the principal section of the crystal, the whole of the reflected light will be polarised perpendicularly to that plane; and therefore if N be *nearly* equal to B, while every thing else remains the same, the reflected pencil will contain some unpolarised light, and will be only partially polarised in a plane perpendicular to the plane of incidence; so that (as Brewster has found by experiment) the crystal will then produce by reflexion the same effect which is produced by ordinary refraction. This (as he also found) will not happen when λ and the angle of incidence are each equal to 45°; because the light is then incident at the polarising angle.

The experiments with oil of cassia must be very difficult, on account of the great feebleness of the reflected light. Sir David Brewster, however, resumed them at different times; and he laid an extensive series of his results before the Physical Section of the British Association at its late meeting in Bristol.

It was not until the latter end of November, 1836, that I became acquainted with the investigations of M. Seebeck, who has contributed greatly to the advancement of the subject. He made very accurate experiments on the light reflected in air from Iceland spar. He detected the deviation, notwithstanding its smallness, and measured it with great care. He also made the first step in the theory of crystalline reflexion; and the remarkable formula (68), which gives the polarising angle when the axis lies in the plane of incidence, is due to him. The hypotheses which he employed were similar to those of Fresnel, and they enabled him to solve the problem of reflexion in the case just mentioned, but not to attempt it generally. The date of his first papers* is the year 1831; but he did not publish his experiments on the deviation until a recent occasion, when he was led to compare them† with the theory which I had originally given in my letter to Sir David Brewster. I have already stated the correction‡ which the theory underwent in consequence of those experiments, and by which it was brought to its present simple form.

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* Poggendorff's Annals, vol. xxi. p. 290; vol. xxii. p. 126.

† Ibid. vol. xxxviii. p. 280.

‡ Two or three months after this correction had been published in the Philosophical Magazine, a notice of it was inserted in Poggendorff's Annals, vol. xl. p. 462. Up to that time, I believe, nothing had been published in Germany on the general theory of crystalline reflexion; at least the writer of the notice (whom I take to be M. Seebeck) does not seem to have heard of any other theory, or any other principles, than mine. But in the next number of Poggendorff, vol. xl. p. 497, there appeared a letter from M. Neumann, in which the writer speaks of a theory of his own, founded on principles exactly the same as those which I had already announced, and refers to a paper which he had communicated on the subject to the Academy of Berlin. The paper has been printed in the Transactions of that Academy for the year 1835; and through the kindness of the author I have received a copy of it, just in time to acknowledge it here. On casting my eye over it, I recognise several equations which are familiar to me; in particular, the equations (vii.), (viii.), (ix.), (x.), which I discovered independently in November last. M. Neumann's paper is very elaborate, and supersedes, in a great measure, the design which I had formed of treating the subject more fully at my leisure; nor can I do better than recommend it to those who wish to pursue the investigations through all their details.

Trinity College, Dublin, *March*, 1838.